

GRADIENT FLOWS OF CLOSED 1-FORMS AND THEIR CLOSED ORBITS

D. SCHÜTZ

ABSTRACT. In [18, 20], Pajitnov considers the closed orbit structure of generic gradient flows of circle-valued Morse functions. It turns out that the torsion of a chain homotopy equivalence between the Novikov complex and the completed simplicial chain complex of the universal cover detects the eta function of the flow. This eta function counts the closed orbits and reduces to the logarithm of the zeta function after abelianizing. We extend this result to the case of closed 1-forms which are Morse. To relate the torsion to the eta function we use the Dennis trace.

1. INTRODUCTION

Given a vector field on a smooth closed manifold M there is a corresponding dynamical system and one can investigate the closed orbits of this flow. It is desirable to collect all closed orbits in one power series and study the algebraic topology and K -theory of this object. To do this observe that closed orbits represent elements in $H_1(M)$ and also in the set of conjugacy classes of $\pi_1(M)$. We set $G = \pi_1(M)$.

In [5], Fried defines a commutative zeta function for certain nonsingular flows as a formal power series and relates it to a Reidemeister torsion invariant of the manifold.

The first noncommutative invariant for flows was introduced in Geoghegan and Nicas [6] for suspension flows. Their analogue of a zeta function is what they call the Lefschetz-Nielsen series which lives in an infinite product of 0-dimensional Hochschild homology groups.

In the case of vector fields with singularities the first papers to obtain relations between zeta functions and torsion are Hutchings and Lee [9, 10] and Pajitnov [18], both dealing with gradients of circle-valued Morse functions and with commutative invariants. In that situation the torsion invariant no longer depends only on the topology of M but the critical points enter via the Novikov complex. Both papers have been generalized, Hutchings [7, 8] discusses closed 1-forms, still in a commutative setting, while Pajitnov [20] gets a noncommutative result for circle-valued Morse functions.

Circle-valued Morse functions correspond to closed rational Morse 1-forms. This paper discusses the noncommutative case for arbitrary closed Morse 1-forms. The geometric methods largely follow Pajitnov [20]. In fact, the geometry in [20] is mainly contained in his earlier paper [18]. The main difficulty is that the algebra required to keep track of the invariants is more complicated than in the commutative case. So instead of looking at a zeta function,

1991 *Mathematics Subject Classification.* Primary 57R70; Secondary 57R25.

Key words and phrases. Novikov complex, closed 1-forms, zeta function, Dennis trace.

Pajitnov [20] and we look at an eta function (or pre-zeta function, compare Fried [5, §2]), which generalizes the logarithm of the zeta function of the commutative case. Since the conjugacy classes of G do not form a group, we cannot take the exponential function of this eta function. To compare this eta function with a certain torsion one needs a logarithm-like homomorphism \mathfrak{L} from K_1 of the Novikov ring to the object containing the eta function. We depart somewhat in the definition of \mathfrak{L} from Pajitnov [20] in that we take a detour through Hochschild homology using the Dennis trace, compare Geoghegan and Nicas [6, §5]. The main theorem we get is

Theorem 1.1. *Let ω be a closed Morse 1-form on a smooth connected closed manifold M^n . Let $\xi : G \rightarrow \mathbb{R}$ be induced by ω and let $C_*^\Delta(\tilde{M})$ be the simplicial $\mathbb{Z}G$ complex coming from a smooth triangulation of M . For every $v \in \mathcal{G}_0(\omega)$ there is a natural chain homotopy equivalence $\varphi(v) : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \rightarrow C_*(\omega, v)$ whose torsion $\tau(\varphi(v))$ lies in \overline{W} and satisfies $\mathfrak{L}(\tau(\varphi(v))) = \eta(-v)$.*

This theorem was obtained by Pajitnov in [20] in the rational case. Here v is the vector field whose eta function we look at, $C_*(\omega, v)$ is the Novikov complex, a complex over the Novikov ring $\widehat{\mathbb{Z}G}_\xi$ and \overline{W} a particular subgroup of $\overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi)$. The set $\mathcal{G}_0(\omega)$ is a set of C^0 -generic vector fields which are gradient with respect to ω , see Section 4. The chain homotopy equivalence can be described as follows: given a smooth triangulation of M , we can adjust this triangulation so that each simplex is transverse to the unstable manifolds of the critical points of ω . Then for a k -simplex σ we define

$$(1) \quad \varphi(v)(\sigma) = \sum_{p \in \text{crit}_k(\omega)} [\sigma : p] p$$

where $\text{crit}_k(\omega)$ is the set of critical points of ω having index k and $[\sigma : p] \in \widehat{\mathbb{Z}G}_\xi$ is the intersection number of a lifting of σ to \tilde{M} with translates of the unstable manifold of a lifting of the critical point p . This chain homotopy equivalence is basically described in Hutchings and Lee [9, §2.3]. The restriction that v lie in $\mathcal{G}_0(\omega)$, a geometric condition due to Pajitnov [18], then allows us to identify the torsion of $\varphi(v)$. This is achieved using the work of Farber and Ranicki [4] and Ranicki [21]. We choose a triangulation of M such that $\varphi(v)$ factors through a complex $\widehat{\mathbb{Z}G}_\xi \otimes C(v)_*$, where $C(v)_*$ is a $\mathbb{Z}G$ complex which comes from a handlebody decomposition on a codimension 1 submanifold N that separates M and a handlebody decomposition on the cobordism obtained by splitting along N . It turns out that the complex $C(v)_*$ is the mapping cone of an injective $\mathbb{Z}G$ homomorphism which depends on the vector field v . After tensoring with the Novikov ring the natural projection to the cokernel is a chain homotopy equivalence. But for $v \in \mathcal{G}_0(\omega)$ the cokernel can be identified with the Novikov complex.

If the Novikov complex is not acyclic the torsion of a chain homotopy equivalence is not determined by the complexes and we will give an example of two ω -gradients v, w with $C_*(\omega, v) = C_*(\omega, w)$, but $\tau(\varphi(v)) \neq \tau(\varphi(w))$, see Remark 5.4.

As mentioned before this paper is closely related to Pajitnov [18, 20]. The work of Hutchings

[7, 8] and Hutchings and Lee [9, 10] is in the same spirit, but with quite different methods. In particular, Hutchings [8] contains a Theorem (see Theorem 5.2 for the precise statement) which might be considered a commutative version of Theorem 1.1. The role of $\tau(\varphi(v))$ is played by two Reidemeister torsions. We show in section 5 how to recover Hutchings' theorem for vector fields in $\mathcal{G}_0(\omega)$ as a corollary of Theorem 1.1. In fact we obtain a stronger "commutative theorem"; see Example 5.3.

The author would like to thank the referee for several suggestions, in particular for formula (1). This paper will form a portion of the author's doctoral dissertation written at the State University of New York at Binghamton under the direction of Ross Geoghegan.

2. MORSE THEORY OF CLOSED 1-FORMS

2.1. Novikov Rings. Let G be a group and $\xi : G \rightarrow \mathbb{R}$ be a homomorphism. For a ring R we denote by $\widehat{\widehat{RG}}$ the abelian group of all functions $G \rightarrow R$. For $\lambda \in \widehat{\widehat{RG}}$ let $\text{supp } \lambda = \{g \in G \mid \lambda(g) \neq 0\}$. Then we define

$$\widehat{RG}_\xi = \{\lambda \in \widehat{\widehat{RG}} \mid \forall r \in \mathbb{R} \quad \#\text{supp } \lambda \cap \xi^{-1}([r, \infty)) < \infty\}$$

For $\lambda_1, \lambda_2 \in \widehat{RG}_\xi$ we set $(\lambda_1 \cdot \lambda_2)(g) = \sum_{\substack{h_1, h_2 \in G \\ h_1 h_2 = g}} \lambda_1(h_1) \lambda_2(h_2)$, then $\lambda_1 \cdot \lambda_2$ is a well defined element of \widehat{RG}_ξ and turns \widehat{RG}_ξ into a ring, the *Novikov ring*. It contains the usual group ring RG as a subring and we have $RG = \widehat{RG}_\xi$ if and only if ξ is the zero homomorphism.

Definition 2.1. The *norm* of $\lambda \in \widehat{RG}_\xi$ is defined to be

$$\|\lambda\| = \|\lambda\|_\xi = \inf\{t \in (0, \infty) \mid \text{supp } \lambda \subset \xi^{-1}((-\infty, \log t])\}$$

It has the following nice properties:

1. $\|\lambda\| \geq 0$ and $\|\lambda\| = 0$ if and only if $\lambda = 0$.
2. $\|\lambda\| = \|\lambda - \mu\|$.
3. $\|\lambda + \mu\| \leq \max\{\|\lambda\|, \|\mu\|\}$.
4. $\|\lambda \cdot \mu\| \leq \|\lambda\| \cdot \|\mu\|$.

If N is a normal subgroup of G that is contained in $\ker \xi$ we get a well defined homomorphism $\bar{\xi} : G/N \rightarrow \mathbb{R}$ and a well defined ring epimorphism $\varepsilon : \widehat{RG}_\xi \rightarrow \widehat{RG}/N_{\bar{\xi}}$ given by $\varepsilon(\lambda)(gN) = \sum_{n \in N} \lambda(gn)$.

Now let Γ be the set of conjugacy classes of G . Again the homomorphism ξ induces a well defined map $\Gamma \rightarrow \mathbb{R}$ which we also denote by ξ . In analogy with above we define $\widehat{R\Gamma}_\xi$, but since there is no well defined multiplication in Γ , this object is just an abelian group. Again there is an epimorphism $\varepsilon : \widehat{RG}_\xi \rightarrow \widehat{R\Gamma}_\xi$ of abelian groups. We can think of $\widehat{R\Gamma}_\xi$ as lying between \widehat{RG}_ξ and $\widehat{RH_1(G)}_{\bar{\xi}}$. If $g \in G$, we denote the conjugacy class of g by $\{g\}$.

Now we will turn our attention to $K_1(\widehat{\mathbb{Z}G}_\xi)$. For the definition of K_1 we refer the reader to Cohen [2] or Milnor [14]. First we disregard units of the form $\pm g$, hence look at $\overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi) =$

$K_1(\widehat{\mathbb{Z}G_\xi})/\langle [\pm g] \rangle$. There is another type of “elementary unit” in $\widehat{\mathbb{Z}G_\xi}$, namely, let $a \in \widehat{\mathbb{Z}G_\xi}$ satisfy $\|a\| < 1$. Then $\sum_{n=0}^{\infty} a^n$ is a well defined element of $\widehat{\mathbb{Z}G_\xi}$ and the inverse of $1 - a$. These form a subgroup of the units in $\widehat{\mathbb{Z}G_\xi}$. We denote the image of this subgroup in $\overline{K}_1^G(\widehat{\mathbb{Z}G_\xi})$ by \overline{W} .

2.2. Closed 1-forms and Vector Fields. Let M^n be a closed connected smooth manifold. By de Rham’s theorem $\{\text{closed 1-forms on } M\}/\{\text{exact 1-forms on } M\} \cong H^1(M; \mathbb{R}) \cong \text{Hom}(H_1(M), \mathbb{R})$, so a closed 1-form ω induces a homomorphism $\xi_\omega : \pi_1(M) \rightarrow \mathbb{R}$ which can be explicitly stated by the formula $\xi_\omega(g) = \int_\gamma \omega \in \mathbb{R}$, where γ is a smooth loop representing $g \in \pi_1(M)$. Set $G = \pi_1(M)$. Then G is finitely presented, so the image of ξ_ω is a finitely generated subgroup of \mathbb{R} , hence isomorphic to \mathbb{Z}^k for some integer k . If $k = 1$ ω is said to be *rational*, if $k > 1$ it is *irrational*.

Rational 1-forms can be described by circle valued functions $f : M \rightarrow S^1$ in the following way: Let $p : \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z} = S^1$ be the usual covering projection, let α be the closed 1-form on S^1 such that $p^*\alpha = dx$; then $f^*\alpha$ is a closed 1-form and $\text{im } \xi_{f^*\alpha} \subset \mathbb{Z} \subset \mathbb{R}$. To obtain other infinite cyclic subgroups of \mathbb{R} as images of ξ one uses circles of different size.

Now, given a rational 1-form ω there is an infinite cyclic covering space $q : \bar{M} \rightarrow M$ such that $q^*\omega = d\bar{f}$, namely the one corresponding to $\ker \xi_\omega$. Let t be the generator of the covering transformation group of \bar{M} with $\bar{f}(tx) > \bar{f}(x)$ for $x \in \bar{M}$. Then \bar{f} defines a map $f : M \rightarrow \mathbb{R}/(\bar{f}(tx) - \bar{f}(x))\mathbb{Z} = S^1$ which induces a surjection on fundamental group.

Notice that for irrational closed 1-forms ω there is a \mathbb{Z}^k -covering space $q : \bar{M} \rightarrow M$ such that $q^*\omega = d\bar{f}$.

Locally a closed 1-form is exact. We will call a closed 1-form a *Morse form* if ω is locally represented by the differential of real valued functions whose critical points are nondegenerate. So if ω is a Morse form, then ω has only finitely many critical points and every critical point has a well defined index.

Definition 2.2. Let ω be a closed 1-form. A vector field v is called an ω -*gradient*, if there exists a Riemannian metric g such that $\omega_x(X) = g(X, v(x))$ for every $x \in M$ and $X \in T_x M$.

The next Lemma allows us to forget about the Riemannian metric and will be useful in using vector fields as gradients of different Morse forms.

Lemma 2.3. *Let ω be a Morse form and v a vector field. Then v is an ω -gradient if and only if*

1. *For every critical point p of ω there exists a neighborhood U_p of p and a Riemannian metric g on U_p such that $\omega_x(X) = g(X, v(x))$ for every $x \in U_p$ and $X \in T_x U_p$.*
2. *If $\omega_x \neq 0$, then $\omega_x(v(x)) > 0$.*

Proof. The “only if” direction is clear. For the “if” direction choose disjoint neighborhoods U_1, \dots, U_k , each with a Riemannian metric coming from 1. for every critical point of ω . Now choose finitely many contractible open sets V_1, \dots, V_m with $\bigcup V_i \subset M - \{\text{critical points}\}$ that together with the U_j ’s cover M . Using 2., it is easy to find a Riemannian metric on each V_i

that turns $v|_{V_i}$ into a gradient of $\omega|_{V_i}$. Now the required Riemannian metric is obtained by using a partition of unity. \square

Remark 2.4. Some authors (e.g. Milnor [13], Pajitnov [16, 18]) use a more restricted version for an ω -gradient, namely, a sharper version of 1. in the Lemma. For an even more general definition of ω -gradient we refer the reader to Pajitnov [19], which contains most of the modifications on a vector field that we will need.

2.3. The Novikov Complex of a Morse form. Given a Morse form ω and an ω -gradient v we denote for a critical point p of ω the unstable, resp. stable, manifold of p by $D_R(p)$, resp. $D_L(p)$. So if $\Phi : M \times \mathbb{R} \rightarrow M$ denotes the flow of v , then $D_R(p) = \{x \in M | \Phi(x, t) \rightarrow p \text{ for } t \rightarrow -\infty\}$ and $D_L(p) = \{x \in M | \Phi(x, t) \rightarrow p \text{ for } t \rightarrow \infty\}$. If the index of p is i , then $D_R(p)$ is an immersed open disk of dimension $n - i$ and $D_L(p)$ of dimension i . We say v satisfies the *transversality condition* if all discs $D_L(p)$ and $D_R(q)$ intersect transversely for all critical points p, q of ω .

Given a Morse form ω and an ω -gradient v satisfying the transversality condition we can define the *Novikov complex* $C_*(\omega, v)$ which is in each dimension i a free $\widehat{\mathbb{Z}G}_\xi$ complex with one generator for every critical point of index i . Here ξ is the homomorphism induced by ω . The boundary homomorphism of $C_*(\omega, v)$ is based on the number of trajectories between critical points of adjacent indices. For more details see Pajitnov [16] or Latour [12]. This chain complex is chain homotopy equivalent to $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M})$, where $C_*^\Delta(\tilde{M})$ is the simplicial chain complex of the universal cover \tilde{M} of M with respect to a smooth triangulation of M lifted to \tilde{M} .¹ Furthermore there is a chain homotopy equivalence whose torsion is in $\overline{W} \subset \overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi)$. In the rational case this is proven in Pajitnov [16], for the general case see Latour [12]. The map described in (1) in the introduction can be used for this. We will show this in 4.1 at least for an ω -gradient v satisfying a “cellularity condition”.

Let us discuss this map. A smooth triangulation Δ of M is called *adjusted to v* , if every k -simplex σ intersects the unstable manifolds $D_R(p)$ transversely for all critical points p of index $\geq k$. To see the existence, assume ω is rational, the general case follows by approximation, see 4.2. A triangulation Δ lifts to a triangulation of \tilde{M} , an infinite cyclic covering space, compare 2.4. For a diffeomorphism ψ of M we denote by $\psi\Delta$ the triangulation of M where simplices are composed with ψ . If we change the triangulation of M by an isotopy, we can get transverse intersections in \tilde{M} of lifted simplices with finitely many unstable manifolds by the results of A.1 in the appendix. Since the results there give openness and density among diffeomorphisms we get a generic set of diffeomorphisms ψ of M isotopic to the identity such that $\psi\Delta$ is adjusted to v .

Given an adjusted triangulation Δ we get a chain map

$$\varphi(v) : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \rightarrow C_*(\omega, v)$$

by formula (1). That $\varphi(v)$ is indeed a chain map follows from the exact case, which is described in the appendix. Lemma A.2 also carries over so that different adjustments to a

¹Of course there are other Novikov complexes corresponding to other regular coverings of M but we are mainly interested in the universal covering.

triangulation lead to chain homotopic maps. Finally, if Δ' is a subdivision of Δ such that $\psi\Delta'$ is adjusted to v , so is $\psi\Delta$ and the diagram

$$\begin{array}{ccc} \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^{\psi\Delta}(\tilde{W}, \tilde{M}_0) & \xrightarrow{\text{sd}} & \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^{\psi\Delta'}(\tilde{W}, \tilde{M}_0) \\ \varphi(v) \searrow & & \swarrow \varphi(v) \\ & C_*(\omega, v) & \end{array}$$

commutes. So once we show that $\varphi(v)$ is a chain homotopy equivalence, its torsion does not depend on the triangulation.

2.4. The Chain Homotopy type of $\cdot \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M})$. The following is a construction of Farber and Ranicki [4] written to fit our purposes, compare also Pajitnov [18, §7].

Given a circle valued Morse map $f : M \rightarrow S^1$ which induces a surjection on fundamental group we get a lifting $\bar{f} : \bar{M} \rightarrow \mathbb{R}$ where \bar{M} is an infinite cyclic covering space. Assuming that $0 \in \mathbb{R}$ is a regular value, set $N = \bar{f}^{-1}(0)$, $M_N = \bar{f}^{-1}([0, 1])$. We get a handle decomposition of the cobordism $(M_N; N, tN = \bar{f}^{-1}(1))$ from the Morse function $\bar{f}|_{M_N} : M_N \rightarrow [0, 1]$. By choosing a cell decomposition of N , Farber and Ranicki [4] construct a finitely generated free $\mathbb{Z}G$ complex $C(v)_*$ homotopy equivalent to $C_*^\Delta(\tilde{M})$. Let us recall the construction from [4, §3]. Let $p : \tilde{M} \rightarrow \bar{M}$ be the universal covering projection and let $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$ be $\bar{f} \circ p$. The cell decomposition of N leads to a cell decomposition of $\tilde{f}^{-1}(\mathbb{Z})$ and the resulting cell complex, denoted by D , is a finitely generated free $\mathbb{Z}G$ complex. Let $c_i(N)$ be the number of i -cells in N . Now E is a finitely generated free $\mathbb{Z}G$ complex containing D as a subcomplex and the remaining generators correspond to critical points of $\tilde{f}|_{M_N}$.

The inclusions $N \hookrightarrow M_N$ and $tN \hookrightarrow M_N$ induce $\mathbb{Z}G$ chain maps $i : D \rightarrow E$ and $k : D \rightarrow E$ and since D is a subcomplex, $i : D \rightarrow E$ is a split injection. Then $C(v)_* = \mathcal{C}(i - k : D \rightarrow E)$ is the mapping cone of $i - k$. In particular $\text{rank } C(v)_i = c_i(N) + c_{i-1}(N) + \# \text{ critical points of } f \text{ having index } i$. In 4.1 we will use the geometry of Pajitnov [18] to get a more detailed version of this chain complex.

Let R be a ring and $\eta : \mathbb{Z}G \rightarrow R$ a ring homomorphism such that $\text{id}_R \otimes_{\mathbb{Z}G} \text{proj}_D(i - k) : R \otimes_{\mathbb{Z}G} D \rightarrow R \otimes_{\mathbb{Z}G} D$ is an automorphism. Then by the Deformation Lemma of Farber and Ranicki [4], see also Ranicki [21, Prop. 1.9], the chain complex $R \otimes_{\mathbb{Z}G} C(v)_*$ is chain homotopy equivalent to $\text{coker}(\text{id}_R \otimes_{\mathbb{Z}G} (i - k)) =: \hat{C}$, a finitely generated free R complex with $\text{rank } \hat{C}_i = \# \text{ critical points of } f \text{ having index } i$. In fact the chain equivalence is identified in [21] to be the natural projection $p : R \otimes_{\mathbb{Z}G} \mathcal{C}(i - k) \rightarrow \text{coker}(\text{id}_R \otimes (i - k))$.

So to use the Deformation Lemma one has to turn a certain square matrix $I - A$ representing $\text{proj}_D(i - k)$ over $\mathbb{Z}G$ into an invertible matrix over a ring R . The matrix A can be chosen to satisfy $\|A_{ij}\|_\xi < 1$ for every entry of A , where ξ is induced by f . Obvious candidates for R are the noncommutative localization used by Farber and Ranicki [4] and the Novikov ring $\widehat{\mathbb{Z}G}_\xi$. A not so obvious candidate is a Novikov ring $\widehat{\mathbb{Z}G}_{\xi'}$ where ξ' is “close” to ξ ; this will be discussed in section 4.

Remark 2.5. Farber [3] has extended the Deformation Lemma of [4] to the case of closed 1-forms using a certain noncommutative localization.

Furthermore, Ranicki [21, Prop. 1.9] contains the calculation of the torsion of the chain homotopy equivalence $p : R \otimes_{\mathbb{Z}G} C(v)_* \rightarrow \hat{C}$. It is given by

$$(2) \quad \tau(p) = \sum_{i=0}^{n-1} (-1)^{i+1} \tau(I - A_i : R \otimes_{\mathbb{Z}G} D_i \rightarrow R \otimes_{\mathbb{Z}G} D_i) \in \overline{K}_1(R).$$

2.5. The Eta Function of a Gradient. Let v be a vector field. By a closed orbit of v we mean a nonconstant map $\gamma : S^1 \rightarrow M$ with $\gamma'(x) = v(\gamma(x))$. The *multiplicity* $m(\gamma)$ is the largest positive integer m such that γ factors through an m -fold covering $S^1 \rightarrow S^1$. We say two closed orbits are *equivalent* if they only differ by a rotation of S^1 . We denote the set of equivalence classes by $Cl(v)$. Notice that $\gamma \in Cl(v)$ gives a well defined element $\{\gamma\} \in \Gamma$. A closed orbit γ is called *nondegenerate* if $\det(I - dP) \neq 0$, where P is a Poincaré map corresponding to γ . In that case we define $\varepsilon(\gamma) \in \{1, -1\}$ to be the sign of $\det(I - dP)$.²

Now let ω be a Morse form. We denote by $\mathcal{G}(\omega)$ the set of all ω -gradients that satisfy the transversality condition and whose closed orbits are nondegenerate. For $v \in \mathcal{G}(\omega)$ we define the *eta-function* of $-v$ to be the element of $\widehat{\mathbb{Q}\Gamma}_\xi$ defined by

$$\eta(-v)(\{g\}) = \sum_{\substack{\gamma \in Cl(-v) \\ \{\gamma\} = \{g\}}} \frac{\varepsilon(\gamma)}{m(\gamma)}$$

Again ξ is induced by ω . For the proof that $\eta(-v)$ is a well defined element of $\widehat{\mathbb{Q}\Gamma}_\xi$ we refer the reader to Hutchings [7, §3.2].

3. ALGEBRAIC CONSTRUCTIONS

3.1. Hochschild Homology. Let R be a ring and S an R -algebra. For an $S - S$ bimodule M we define the *Hochschild chain complex* $(C_*(S, M), d)$ by $C_n(S, M) = S \otimes \dots \otimes S \otimes M$, where the product contains n copies of S and the tensor products are taken over R . The boundary operator is given by

$$\begin{aligned} d(s_1 \otimes \dots \otimes s_n \otimes m) &= s_2 \otimes \dots \otimes s_n \otimes m s_1 \\ &\quad + \sum_{i=1}^{n-1} (-1)^i s_1 \otimes \dots \otimes s_i s_{i+1} \otimes \dots \otimes s_n \otimes m \\ &\quad + (-1)^n s_1 \otimes \dots \otimes s_{n-1} \otimes s_n m \end{aligned}$$

The n -th Hochschild homology group of S with coefficients in M is denoted by $HH_n(S, M)$. If $M = S$ and the bimodule structure is given by ordinary multiplication we write $HH_*(S)$ instead of $HH_*(S, M)$. We will mainly be interested in the case where $R = \mathbb{Z}$, $S = M = \widehat{\mathbb{Z}G}_\xi$ is a Novikov ring and $n = 1$. A useful observation is that $d(1 \otimes 1 \otimes x) = 1 \otimes x$ and hence classes represented by $1 \otimes x$ are automatically 0 in $HH_1(S, M)$.

²i.e. $\varepsilon(\gamma)$ is the fixed point index of P at the isolated fixed point coming from the closed orbit.

Given an $n \times k$ matrix $A = (A_{ij})$ over S and an $k \times n$ matrix $B = (B_{ij})$ over M we define an $n \times n$ matrix $A \otimes B$ with entries in $S \otimes M$ by setting $(A \otimes B)_{ij} = \sum_{l=1}^k A_{il} \otimes B_{lj}$. The trace of this matrix, $\text{trace } A \otimes B$, is given by $\sum_{l,m} A_{lm} \otimes B_{ml}$ and is an element of $C_1(S, M)$, it is a cycle if and only if $\text{trace}(AB) = \text{trace}(BA)$. For more on Hochschild homology see Geoghegan and Nicas [6] or Igusa [11].

3.2. The homomorphism \mathfrak{L} . For a ring R with unit, which, in view of subsection 3.1, we can think of as a \mathbb{Z} -algebra, there is the Dennis trace homomorphism $DT : K_1(R) \rightarrow HH_1(R)$ defined as follows: If $\alpha \in K_1(R)$ is represented by the matrix A , then $DT(\alpha) = [\text{trace } (A \otimes A^{-1})] \in HH_1(R)$, see Igusa [11, §1]. It is easy to see that the Dennis trace factors through $\overline{K}_1(R) = K_1(R)/\langle [-1] \rangle$.

We want to define a homomorphism $\mathfrak{L} : \overline{W} \rightarrow \widehat{\mathbb{Q}\Gamma}_\xi$. To do this define $m : C_1(\widehat{\mathbb{Z}G}_\xi, \widehat{\mathbb{Z}G}_\xi) = \widehat{\mathbb{Z}G}_\xi \otimes \widehat{\mathbb{Z}G}_\xi \rightarrow \widehat{\mathbb{R}\Gamma}_\xi$ by

$$m(\lambda_1 \otimes \lambda_2) : \gamma \mapsto \begin{cases} \sum_{\substack{h_1, h_2 \in G \\ \{h_1 h_2\} = \gamma}} \frac{\xi(h_1)}{\xi(\gamma)} \lambda_1(h_1) \lambda_2(h_2) & \text{if } \xi(\gamma) < 0 \\ 0 & \text{if } \xi(\gamma) \geq 0 \end{cases}$$

We can think of m as a weighted combination of multiplication in $\widehat{\mathbb{Z}G}_\xi$ and the augmentation $\varepsilon : \widehat{\mathbb{Z}G}_\xi \rightarrow \widehat{\mathbb{Z}\Gamma}_\xi$.

Lemma 3.1. *The homomorphism m induces a homomorphism $\mu : HH_1(\widehat{\mathbb{Z}G}_\xi) \rightarrow \widehat{\mathbb{R}\Gamma}_\xi$.*

Proof. It is to be shown that m vanishes on boundaries. So let $\gamma \in \Gamma$ satisfy $\xi(\gamma) < 0$, then

$$\begin{aligned} & m(\lambda_2 \otimes \lambda_3 \lambda_1 - \lambda_1 \lambda_2 \otimes \lambda_3 + \lambda_1 \otimes \lambda_2 \lambda_3)(\gamma) \\ &= \sum_{\substack{g, h \in G \\ \{gh\} = \gamma}} \frac{\xi(g)}{\xi(\gamma)} \lambda_2(g) \lambda_3 \lambda_1(h) - \sum_{\substack{g, h \in G \\ \{gh\} = \gamma}} \frac{\xi(g)}{\xi(\gamma)} \lambda_1 \lambda_2(g) \lambda_3(h) + \sum_{\substack{g, h \in G \\ \{gh\} = \gamma}} \frac{\xi(g)}{\xi(\gamma)} \lambda_1(g) \lambda_2 \lambda_3(h) \\ &= \frac{1}{\xi(\gamma)} \left(\sum_{\substack{g_2, g_3, g_1 \in G \\ \{g_2 g_3 g_1\} = \gamma}} \xi(g_2) \lambda_2(g_2) \lambda_3(g_3) \lambda_1(g_1) - \sum_{\substack{g_1, g_2, g_3 \in G \\ \{g_1 g_2 g_3\} = \gamma}} \xi(g_1 g_2) \lambda_1(g_1) \lambda_2(g_2) \lambda_3(g_3) \right. \\ &\quad \left. + \sum_{\substack{g_1, g_2, g_3 \in G \\ \{g_1 g_2 g_3\} = \gamma}} \xi(g_1) \lambda_1(g_1) \lambda_2(g_2) \lambda_3(g_3) \right) \\ &= 0 \end{aligned}$$

since $\{g_2 g_3 g_1\} = \{g_1 g_2 g_3\}$ and ξ is a homomorphism. \square

For $g \in G$ we have $m(g \otimes g^{-1}) = 0$, therefore the composition $\mu \circ DT$ factors through $\overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi)$, call this homomorphism $L : \overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi) \rightarrow \widehat{\mathbb{R}\Gamma}_\xi$. We want to examine how this homomorphism behaves on \overline{W} . For future purposes it will be useful not just to look at 1×1 matrices.

Definition 3.2. An $n \times n$ matrix A over $\widehat{\mathbb{Z}G}_\xi$ is called ξ -regular, if there exists $K < 0$ such that $\|A_{i_1 i_2}\| \cdot \|A_{i_2 i_3}\| \cdots \|A_{i_m i_1}\| \leq \exp(Km)$ for all $m \geq 1$, $1 \leq i_j \leq n$, $1 \leq j \leq m$.

For example, a matrix A which satisfies $\|A_{ij}\| < 1$ for every entry is ξ -regular, but ξ -regular matrices can have entries A_{ij} with $\|A_{ij}\| \geq 1$. The motivation for ξ -regular matrices comes from our approximation arguments, see Remark 4.4.

Lemma 3.3. *Let A be a ξ -regular matrix. Then*

1. *The matrix $I - A$ is invertible over $\widehat{\mathbb{Z}G}_\xi$ and the inverse is given by $I + A + A^2 + \dots$.*
2. *Denote the image of $I - A$ in $\overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi)$ by $\tau(I - A)$; then $\tau(I - A) \in \overline{W}$.*

Proof. 1. We need to show that $I + A + A^2 + \dots$ is a well defined matrix over $\widehat{\mathbb{Z}G}_\xi$. Note

that $(A^m)_{ij} = \sum_{i_1, \dots, i_{m-1}=1}^n A_{ii_1} A_{i_1 i_2} \cdots A_{i_{m-1} j}$. We will look at terms of the form

$A_* = A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_{m-1} i_m}$ and get an estimate for $\|A_*\|$. The idea is to write A_* as a word $C_1 D_1 \cdots C_l D_l$ where the length of the word $C_1 \cdots C_l$ is smaller than n and the words D_j are of the form $A_{j_1 j_2} \cdots A_{j_k j_1}$.

So assume that $A_* = A_{i_1 i_2} A_{i_2 i_3} \cdots A_{i_{m-1} i_m}$ where $m > n + 1$. Let i_j be the first index whose value appears more than once. Since these numbers are between 1 and n we have $j \leq n$. Let k be the largest number such that $i_k = i_j$, then

$$A_* = A_{i_1 i_2} \cdots A_{i_{j-1} i_j} A_{i_j i_{j+1}} \cdots A_{i_{k-1} i_k} A_{i_k i_{k+1}} \cdots A_{i_{m-1} i_m} = B_1 \cdots B_{j-1} D_1 B_j A_{1*}$$

where $B_1 = A_{i_1 i_2}, \dots, B_{j-1} = A_{i_{j-1} i_j}$, $D_1 = A_{i_j i_{j+1}} \cdots A_{i_{k-1} i_k}$, $B_j = A_{i_k i_{k+1}}$ and $A_{1*} = A_{i_{k+1} i_{k+2}} \cdots A_{i_{m-1} i_m}$. Notice that $\|D_1\| \leq \exp(K(k-j))$. Now look at A_{1*} ; among the indices i_{k+1}, \dots, i_{m-1} are at most $n-j$ numbers; the numbers i_1, \dots, i_j , which are all different, do not appear. If $m-1-k > n-j$, one of these numbers will appear more than once. Let i_{k+j_1} be the first such index and i_{k+k_1} the last index equal to i_{k+j_1} . Again we get $j_1 \leq n-j$, hence $j+j_1 \leq n$. As above we can write

$$A_* = B_1 \cdots B_{j-1} D_1 B_j \cdots B_{j+j_1-1} D_2 B_{j+j_1} A_{2*}$$

We continue this process until we get

$$A_* = B_1 \cdots B_{j-1} D_1 B_j \cdots B_{j+j_1-1} D_2 B_{j+j_1} \cdots D_r \cdots B_l$$

with $l \leq n$. Since A_* consists of $m-1$ letters we get $\|D_1\| \cdots \|D_r\| \leq \exp(K(m-1-n))$. Let $M \in \mathbb{R}$ be a number such that $\|A_{st}\| \leq M$ for all s, t . Then $\|A_*\| \leq \exp(K(m-1-n)) \cdot M^n$.

Since $K(m-1-n) \rightarrow -\infty$ as $m \rightarrow \infty$, $\sum_{m=0}^{\infty} A_{ij}^m$ is a well defined element of $\widehat{\mathbb{Z}G}_\xi$.

2. The argument is the same as in Pajitnov [20, Prop. 1.2]. Using elementary row reductions we obtain a matrix of the form

$$\begin{pmatrix} 1 - A_{11} & -A_{12} & \cdots & -A_{1n} \\ 0 & & & \\ \vdots & & I - A' & \\ 0 & & & \end{pmatrix}$$

where A' is an $n - 1 \times n - 1$ matrix which is again ξ -regular with the same K . Induction gives the result. \square

Proposition 3.4. *Let A be a ξ -regular matrix over $\widehat{\mathbb{Z}G}_\xi$, then*

$$L(\tau(I - A)) = -\varepsilon \left(\sum_{m=1}^{\infty} \frac{\text{trace } A^m}{m} \right)$$

In particular, L induces a homomorphism $\mathfrak{L} : \overline{W} \rightarrow \widehat{\mathbb{Q}\Gamma}_\xi$.

Proof. As before denote the image of $I - A$ in $\overline{K}_1(\widehat{\mathbb{Z}G}_\xi)$ by $\tau(I - A)$. Then

$$\begin{aligned} L(\tau(I - A)) &= \mu \circ DT(\tau(I - A)) = \mu [\text{trace } (I - A \otimes \sum_{m=0}^{\infty} A^m)] \\ &= -\mu [\text{trace } (A \otimes \sum_{m=0}^{\infty} A^m)] \quad \text{since } 1 \otimes x \text{ is a boundary} \\ &= -\sum_{m=0}^{\infty} \sum_{i,k=1}^n \mu [A_{ik} \otimes A_{ki}^m]. \end{aligned}$$

It is sufficient to show that

$$(3) \quad \sum_{i,k=1}^n \mu [A_{ik} \otimes A_{ki}^m] = \varepsilon \left(\frac{\text{trace } A^{m+1}}{m+1} \right)$$

Both sides are clearly 0 for $\gamma \in \Gamma$ with $\xi(\gamma) \geq 0$. Call the left side of (3) X and let $\gamma \in \Gamma$ with $\xi(\gamma) < 0$. Then

$$X(\gamma) = \frac{1}{\xi(\gamma)} \sum_{i_1, \dots, i_{m+1}=1}^n \sum_{\substack{h_1, \dots, h_{m+1} \in G \\ \{h_1 \dots h_{m+1}\} = \gamma}} \xi(h_1) A_{i_1 i_2}(h_1) \cdots A_{i_{m+1} i_1}(h_{m+1}).$$

Let \mathbb{Z}_{m+1} act on $\{1, \dots, m+1\}$ by the cycle $(1 \ 2 \ \dots \ m+1)$ and on $\{1, \dots, n\}^{m+1}$ by rotation. For $x \in \{1, \dots, n\}^{m+1}$ denote by $[x]$ the orbit of x and by S the orbit set. We get

$$\begin{aligned} X(\gamma) &= \frac{1}{\xi(\gamma)} \sum_{i_1, \dots, i_{m+1}=1}^n \sum_{\substack{h_1, \dots, h_{m+1} \in G \\ \{h_1 \dots h_{m+1}\} = \gamma}} \frac{1}{m+1} \sum_{t \in \mathbb{Z}_{m+1}} \xi(h_{t1}) A_{i_1 i_2}(h_{t1}) \cdots A_{i_{m+1} i_1}(h_{t(m+1)}) \\ &= \frac{1}{m+1} \frac{1}{\xi(\gamma)} \sum_{[x] \in S} \sum_{(i_1, \dots, i_{m+1}) \in [x]} \sum_{\substack{h_1, \dots, h_{m+1} \in G \\ \{h_1 \dots h_{m+1}\} = \gamma}} \sum_{t \in \mathbb{Z}_{m+1}} \xi(h_{t1}) A_{i_1 i_2}(h_{t1}) \cdots A_{i_{m+1} i_1}(h_{t(m+1)}) \\ &= \frac{1}{m+1} \frac{1}{\xi(\gamma)} \sum_{[x] \in S} \sum_{\substack{h_1, \dots, h_{m+1} \in G \\ \{h_1 \dots h_{m+1}\} = \gamma}} \sum_{t \in \mathbb{Z}_{m+1}} \xi(h_{t1}) \sum_{(i_1, \dots, i_{m+1}) \in [x]} A_{i_1 i_2}(h_{t1}) \cdots A_{i_{m+1} i_1}(h_{t(m+1)}) \end{aligned}$$

Now

$$\sum_{(i_1, \dots, i_{m+1}) \in [x]} A_{i_1 i_2}(h_{t1}) \cdots A_{i_{m+1} i_1}(h_{t(m+1)}) = \sum_{(i_1, \dots, i_{m+1}) \in [x]} A_{i_1 i_2}(h_1) \cdots A_{i_{m+1} i_1}(h_{m+1}),$$

since the orbit is obtained by shifting (i_1, \dots, i_{m+1}) , so

$$\begin{aligned} X(\gamma) &= \frac{1}{m+1} \frac{1}{\xi(\gamma)} \sum_{[x] \in S} \sum_{\substack{h_1, \dots, h_{m+1} \in G \\ \{h_1 \cdots h_{m+1}\} = \gamma}} \sum_{(i_1, \dots, i_{m+1}) \in [x]} A_{i_1 i_2}(h_1) \cdots A_{i_{m+1} i_1}(h_{m+1}) \sum_{t \in \mathbb{Z}_{m+1}} \xi(h_{t1}) \\ &= \frac{1}{m+1} \frac{\xi(\gamma)}{\xi(\gamma)} \sum_{[x] \in S} \sum_{(i_1, \dots, i_{m+1}) \in [x]} \sum_{\substack{h_1, \dots, h_{m+1} \in G \\ \{h_1 \cdots h_{m+1}\} = \gamma}} A_{i_1 i_2}(h_1) \cdots A_{i_{m+1} i_1}(h_{m+1}) \\ &= \frac{1}{m+1} \sum_{i_1, \dots, i_{m+1}=1}^n \sum_{\substack{h_1, \dots, h_{m+1} \in G \\ \{h_1 \cdots h_{m+1}\} = \gamma}} A_{i_1 i_2}(h_1) \cdots A_{i_{m+1} i_1}(h_{m+1}) \\ &= \frac{1}{m+1} \varepsilon(\text{trace } A^{m+1})(\gamma). \end{aligned}$$

□

In the case where ξ is a homomorphism to the integers it is now easily seen that \mathfrak{L} agrees with Pajitnov's \mathfrak{L} from [20] on \overline{W} once the correct identifications are made.

4. GEOMETRY OF MORSE FORMS

4.1. The chain homotopy equivalence. In [18, 20], Pajitnov defines a condition (\mathfrak{C}') , in [19] denoted by (\mathfrak{CC}) , for an f -gradient v , where $f : M \rightarrow S^1$ is a Morse function that induces a surjection ξ on fundamental group. For the full condition we refer the reader to these papers, but informally, it can be described as follows: just as in 2.4 we get a cobordism (M_N, N, tN) and a Morse map $\bar{f} : (M_N, N, tN) \rightarrow ([0, b], \{0\}, \{b\})$. Here we use an arbitrary $b > 0$ instead of just $b = 1$ to indicate that f might come from a rational Morse form. Now the condition (\mathfrak{C}') requires a Morse map ψ on N and a ψ -gradient u which gives a handle decomposition on N and tN . The vector field v which lifts to a vector field v' on M_N now has to satisfy a “cellularity condition”: whenever p is a critical point of \bar{f} of index i , it should be the case that some thickening of $D_L(p)$ is attached to the union of the $(i-1)$ -handles of N and of M_N . Also a thickening of an i -handle in tN has to flow under $-v'$ into the i -skeleton of N and M_N . A symmetric condition holds for right hand discs and handles of N .

By Pajitnov [19, Prop.5.4], the set of f -gradients satisfying (\mathfrak{C}') and the transversality condition is C^0 -open and dense in the set of f -gradients that satisfy the transversality condition. Such gradients should be thought of as cellular approximations of arbitrary f -gradients.

Now let $\rho : \tilde{M} \rightarrow M$ be the universal cover, $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$ the lifting of f and $\tilde{M}_N = \tilde{f}^{-1}([0, b])$. As in A.2 we get a filtration of M_N which we denote by M_i so that $C_i^{MS}(\tilde{M}_N) = H_i(\tilde{M}_i, \tilde{M}_{i-1})$ gives a free $\mathbb{Z}H$ complex calculating $H_*(\tilde{M}_N)$, where $H = \ker \xi$. We also have $C_i^{MS}(\tilde{M}_N) = C_i^{MS}(\tilde{N}) \oplus C_i^{MS}(\tilde{M}_N, \tilde{N})$. Denote $C_*^{MS}(\tilde{M}_N; G) = \mathbb{Z}G \otimes_{\mathbb{Z}H} C_*^{MS}(\tilde{M}_N)$ and similarly for other $\mathbb{Z}H$ complexes.

The flow of $-v$ on M_N induces a chain map

$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} : C_*^{MS}(\tilde{N}; G) \rightarrow C_*^{MS}(\tilde{N}; G) \oplus C_*^{MS}(\tilde{M}_N, \tilde{N}; G) = C_*^{MS}(\tilde{M}_N; G)$ by starting in $C_\delta^k(u) \subset t\tilde{N}$ and flowing into $(\tilde{M}_N)_i$. See appendix A.2 for the sets $C_\delta(\psi)$ and details on this flowing. Notice that $\delta > 0$ is given through condition (\mathfrak{C}') . The map k_1 is the homological gradient descent of Pajitnov [18, §4]. Choose a basis of $C_*^{MS}(\tilde{N}; G)$ by lifting critical points of ψ and let the matrix A_i represent the homomorphism k_1 in dimension i . Since we can choose the liftings within \tilde{N} , we get $\|A_{jk}\| < 1$ for the entries of every matrix A_i .

Proposition 4.1. *Let v be an f -gradient satisfying the transversality condition and (\mathfrak{C}') . Then $\varphi(v)$ is a chain homotopy equivalence and*

$$\tau(\varphi(v)) = \sum_{i=0}^{n-1} (-1)^{i+1} \tau(I - A_i) \in \overline{K}_1^G(\widehat{\mathbb{Z}G_\xi}).$$

Proof. Let Δ be a triangulation of M which has N as a subcomplex Δ' . This induces a triangulation Δ^c of M_N which has two copies of Δ' as subcomplexes. Denote the one corresponding to N by Δ_0 and the one corresponding to tN by Δ_1 . Assume Δ has the following properties:

1. Δ is adjusted to v .
2. Δ' is adjusted to u .
3. There is an $\varepsilon > \delta$ such that if σ is a k -simplex in Δ' , then $\sigma \subset C_\varepsilon^k(u)$.
4. There is an $\varepsilon > \delta$ such that if $x \in M_N^{(k)}$ and the trajectory of $-v$ starting at x ends in $fl(x) \in N$, then $fl(x) \in C_\varepsilon^k(u)$.

The existence of such a triangulation is shown in A.3.

A.2 gives a simple chain homotopy equivalence

$$\varphi : C_*^\Delta(\tilde{M}_N; G) \rightarrow C_*^{MS}(\tilde{M}_N; G).$$

Let us define a chain map $s : C_*^\Delta(\tilde{N}; G) \rightarrow C_*^\Delta(\tilde{M}_N, \tilde{N}; G) \subset C_*^\Delta(\tilde{M}_N; G)$. If σ is a simplex in Δ' , look at the liftings $\sigma_0 \subset \tilde{N}$ and $\sigma_1 \subset t\tilde{N}$ used for the basis of $C_*^\Delta(\tilde{M}_N)$. There is exactly one $g \in G$ such that $g\sigma_1 = \sigma_0$ in \tilde{M} . Set $s(\sigma_0) = g\sigma_1$. Look at the diagram

$$\begin{array}{ccc} C_*^\Delta(\tilde{N}; G) & \xrightarrow{s} & C_*^\Delta(\tilde{M}_N; G) \\ \downarrow \varphi_1 & & \downarrow \varphi \\ C_*^{MS}(\tilde{N}; G) & \xrightarrow{k} & C_*^{MS}(\tilde{M}_N; G) \end{array}$$

Because of property 4. above, this diagram commutes. Therefore the map $\begin{pmatrix} \varphi & 0 \\ 0 & \varphi_1 \end{pmatrix}$ is a simple homotopy equivalence between the mapping cones $\mathcal{C}(i - s)$ and $\mathcal{C}(i - k)$, where i represents inclusion. But by the Deformation Lemma of Farber and Ranicki [4], $\mathcal{C}(i - s)$ is chain homotopy equivalent to $\text{coker}(i - s)$, in fact simple homotopy equivalent by Ranicki [21, Prop.1.9] (the corresponding matrix term is just I). But $\text{coker}(i - s)$ is easily seen to be $C_*^\Delta(\tilde{M})$.

After tensoring with the Novikov ring we have the following sequence of chain homotopy equivalences

$$\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \rightarrow \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} \mathcal{C}(i-s) \rightarrow \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} \mathcal{C}(i-k) \rightarrow \text{coker}(\text{id}_{\widehat{\mathbb{Z}G}_\xi} \otimes i-k)$$

and all except the last one are simple. The first map is described in the proof of Ranicki [21, Prop.1.7]. Because of the special form of the vector field v the Novikov complex $C_*(f, v)$ can be identified with $\text{coker}(\text{id}_{\widehat{\mathbb{Z}G}_\xi} \otimes i-k)$, see Ranicki [21, Remark 4.8] and Pajitnov [18, Remark 7.3]. We claim that this composition is exactly $\varphi(v)$. Denote the composition by θ .

We denote $t^k \tilde{M}_N = \tilde{f}^{-1}[bk, b(k+1)]$. Let $\sigma \in C_k^\Delta(\tilde{M})$, lift it to $\bar{\sigma} \in C_k^\Delta(\tilde{M}_N, \tilde{N}; G)$ (if σ is a cell in \tilde{N} , lift it to $\bar{\sigma} \subset t\tilde{N}$). Then

$$\theta(\sigma) = [\varphi(0, \bar{\sigma})] = [\varphi_N(\bar{\sigma}), \varphi_r(\bar{\sigma})] \in \text{coker}(i-k) = C_*(\omega, v),$$

where $\varphi_N(\bar{\sigma}) \in C_k^\Delta(\tilde{N}; G)$ and $\varphi_r(\bar{\sigma}) \in C_k^\Delta(\tilde{M}_N, \tilde{N}; G)$ are defined to give $(\varphi_N(\bar{\sigma}), \varphi_r(\bar{\sigma})) = \varphi(0, \bar{\sigma})$.

Now $[\varphi_N(\bar{\sigma}), \varphi_r(\bar{\sigma})] = [\varphi_N(\bar{\sigma}), 0] + [0, \varphi_r(\bar{\sigma})]$. $\varphi_r(\bar{\sigma})$ represents the part of $\bar{\sigma}$ that flows into critical points of index k in \tilde{M}_N under $-v$ while $\varphi_N(\bar{\sigma})$ represents the part that flows into \tilde{N} . Now

$$\begin{aligned} (i-k)\left(\sum_{m=0}^{\infty} k_1^m(x)\right) &= \left(\sum_{m=0}^{\infty} k_1^m(x), 0\right) - \left(\sum_{m=0}^{\infty} k_1^{m+1}(x), k_2\left(\sum_{m=0}^{\infty} k_1^m(x)\right)\right) \\ &= (x, k_2\left(\sum_{m=0}^{\infty} k_1^m(x)\right)). \end{aligned}$$

With $x = \varphi_N(\bar{\sigma})$ we thus get

$$[\varphi_N(\bar{\sigma}), 0] = [0, k_2\left(\sum_{m=0}^{\infty} k_1^m(\varphi_N(\bar{\sigma}))\right)].$$

But $k_2(k_1^m(\varphi_N(\bar{\sigma})))$ represents the part of $\bar{\sigma}$ that flows into critical points of index k in $t^{-m-1}\tilde{M}_N$ under $-v$. Therefore $\theta = \varphi(v)$ and $\varphi(v)$ is a chain homotopy equivalence whose torsion is given by (2). \square

Remark 4.2. Pajitnov [18, 20] obtains a chain homotopy equivalence

$\psi(v) : C_*(f, v) \rightarrow \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M})$ by including the Novikov complex into a complex C' simple homotopy equivalent to $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} \mathcal{C}(i-k)$, in fact the map from C' to $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} \mathcal{C}(i-k)$ is just a simple change of basis, compare [18, §7.4]. The composition of this equivalence with Ranicki's equivalence $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} \mathcal{C}(i-k) \rightarrow C_*(f, v)$ is readily seen to be the identity on $C_*(f, v)$.

4.2. Approximation of irrational forms by rational forms. In this subsection we describe a useful method due to Pajitnov [17, §2B]. Given a Morse form ω and an ω -gradient v , the induced homomorphism $\bar{\xi}_\omega : H_1(M) \rightarrow \mathbb{R}$ splits $H_1(M)$ as $\mathbb{Z}^k \oplus \ker \bar{\xi}_\omega$. Choose $g_1, \dots, g_k \in G$ so that the images $\bar{g}_1, \dots, \bar{g}_k \in H_1(M)$ generate the \mathbb{Z}^k part. Now let $\omega_1, \dots, \omega_k$ be closed 1-forms with $\bar{\xi}_{\omega_j}(\bar{g}_i) = \delta_{ji}$ and $\bar{\xi}_{\omega_j}$ vanishes on $\ker \bar{\xi}_\omega$. Then $\bar{\xi}_{\omega_j} : G \rightarrow \mathbb{Z}$ vanishes

on $\ker \xi_\omega$ and satisfies $\xi_{\omega_j}(g_i) = \delta_{ji}$. Furthermore the closed 1-forms can be chosen to vanish in a neighborhood of the critical points of ω .

For $x \in \mathbb{R}^k$ we can now define $\omega_x = \omega + \sum_{j=1}^k x_j \omega_j$. By choosing the x_j small we can make sure that the ω -gradient v is also an ω_x -gradient. To see this notice that in a neighborhood of the critical points of ω the new form agrees with ω . Denote the complement of this neighborhood by C . Because of the compactness of C and Lemma 2.3 there is a $K > 0$ such that $\omega_p(v(p)) \geq K$ for all $p \in C$. Now the x_j have to be chosen so small that $(\omega_x)_p(v(p)) > 0$ for all $p \in C$ which is possible again by compactness. Lemma 2.3 now gives that v is an ω_x -gradient.

We have $\xi_{\omega_x}(g_j) = \xi_\omega(g_j) + x_j \xi_{\omega_j}(g_j)$, so we can also choose the x_j to have $\xi_{\omega_x} : G \rightarrow \mathbb{R}$ factor through \mathbb{Q} . Hence we get

Lemma 4.3. *For a Morse form ω and an ω -gradient v there exists a rational Morse form ω' with the same set of critical points and that agrees with ω in a neighborhood of these critical points such that v is also an ω' -gradient.*

Let us compare the Novikov complexes we obtain for a Morse form ω and a rational approximation ω' that both use the same vector field v . The complexes are taken over different rings, $\widehat{\mathbb{Z}G}_{\xi_\omega}$ and $\widehat{\mathbb{Z}G}_{\xi_{\omega'}}$, respectively. But for two critical points p, q of adjacent index the elements $\tilde{\partial}(p, q) \in \widehat{\mathbb{Z}G}_{\xi_\omega}$ and $\tilde{\partial}'(p, q) \in \widehat{\mathbb{Z}G}_{\xi_{\omega'}}$ agree when viewed as elements of $\widehat{\mathbb{Z}G}$ since both count the number of flowlines between \tilde{p} and translates of \tilde{q} , and these only depend on v . So we can compare chain complexes even though they are over different rings. This is an important observation and will remain useful in the next subsection.

4.3. Comparison of the eta function with torsion. Again let ω be a Morse form. An ω -gradient v satisfies the condition $(\mathfrak{A}\mathfrak{C})$, if there exists a rational Morse form ω' such that v is an ω' -gradient and as such it satisfies (\mathfrak{C}') . We think of this condition as “approximately (\mathfrak{C}') ”.

We want to carry over the density results of Pajitnov [19]. Then C^0 -density in $\mathcal{G}(\omega)$ can be seen as follows: given an ω -gradient v' there is by Lemma 4.3 a rational Morse form ω' that agrees with ω near the critical points and such that v' is also an ω' -gradient. Now the density of ω' -gradients satisfying (\mathfrak{C}') allows us to choose a vector field v as close as we like to v' . To see that we can find an ω -gradient this way observe that 1. of Lemma 2.3 is trivially fulfilled and since $\omega(v') \geq K > 0$ away from a neighborhood of the critical points we get $\omega(v) > 0$ by choosing v close enough to v' . Therefore v is an ω gradient satisfying $(\mathfrak{A}\mathfrak{C})$. The C^0 -openness now follows from Pajitnov [19, Prop.5.4].

Now if an ω -gradient v satisfies $(\mathfrak{A}\mathfrak{C})$, let ω' be the rational Morse form as in the definition and denote by $\xi : G \rightarrow \mathbb{R}$ and $\xi' : G \rightarrow \mathbb{R}$ the homomorphisms induced by ω and ω' . For the rational form ω' we can form \tilde{N} , the $\mathbb{Z}G$ complex $\mathcal{C}(i - k)$, the homomorphism $k_1 : C_i^{MS}(\tilde{N}; G) \rightarrow C_i^{MS}(\tilde{N}; G)$ and the matrix A_i just as in 4.1 and the proof of Proposition 4.1. If we can show that $I - A_i$ is invertible over $\widehat{\mathbb{Z}G}_\xi$ we get the chain homotopy equivalence between $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} \mathcal{C}(i - k)$ and $\text{coker}(\text{id} \otimes i - k)$. We know from 4.1 that $I - A_i$ is invertible over

$\widehat{\mathbb{Z}G}_{\xi'}$ and that the cokernel over this Novikov ring is exactly the Novikov complex $C_*(\omega', v)$. In order to keep the notation simple denote the matrix A_i by B .

Remark 4.4. That $I - B$ is invertible over $\widehat{\mathbb{Z}G}_{\xi'}$ is easily seen since a basis can be chosen so that $\|B_{ij}\|_{\xi'} < 1$ for every entry B_{ij} . If we could choose a basis of D such that $\|B_{ij}\|_{\xi} < 1$ we would also immediately get that $I - B$ is invertible over $\widehat{\mathbb{Z}G}_{\xi}$. A similar argument is used in Latour [12, §2.23]. But since we have to choose liftings of cells instead of critical points it is not clear that a nice basis can be chosen. So instead of trying to find a nice basis we use the notion of ξ -regular matrices.

Proposition 4.5. *The matrix B over $\mathbb{Z}G$ is ξ -regular.*

Proof. We have chosen a basis of D_i by choosing i -cells in \tilde{N} , call these cells σ_k . If $h \in \text{supp } B_{jk}$, then there exist negative flowlines from σ_j to $h\sigma_k$ by the construction of B .

We need to show that there exists a $K < 0$ with the property that given $m \geq 1$ and indices j, n_1, \dots, n_{m-1} and $g_1 \in \text{supp } B_{jn_1}$, $g_2 \in \text{supp } B_{n_1n_2}, \dots, g_m \in \text{supp } B_{n_{m-1}j}$ we have $\xi(g_1 \cdots g_m) \leq K \cdot m$.

Now we have to recall the proof of the Main Theorem in Pajitnov [20, §5]. Every cell σ_k defines a thickened sphere in $\coprod_{l \in \mathbb{Z}} \tilde{V}_l^{[i]} / \tilde{V}_l^{(i-1)}$ that we denote by s_k , also let $g = g_1 \cdots g_m$. The

spaces $\tilde{V}_l^{[i]}$ and $\tilde{V}_l^{(i)}$ are defined in Pajitnov [20, §4.4, §4.5]. Since $g_1 \in \text{supp } B_{jn_1}$ we have that $-\tilde{v}$ induces a homologically nontrivial map from s_j to $g_1 s_{n_1}$. Similarly every $g_l \in \text{supp } B_{n_{l-1}n_l}$ gives rise to a homologically nontrivial map from $g_1 \cdots g_{l-1} s_{n_{l-1}}$ to $g_1 \cdots g_l s_{n_l}$. The composition of all these maps plus $g^{-1} : g s_j \rightarrow s_j$ is homologically nontrivial and hence has a fixed point other than the base point, compare the proof of Lemma 5.1 in [20]. Notice that the existence of this fixed point does not require the condition that closed orbits of v are nondegenerate. This fixed point corresponds to a flow line $\gamma : [a_1, a_2] \rightarrow \tilde{M}$ of $-\tilde{v}$ with $\gamma(a_1) = x \in \sigma_j$ and $\gamma(a_2) = gx \in g\sigma_j$ which passes through the cells $g_1 \cdots g_l \sigma_{n_l}$.

We need the following

Lemma 4.6. *There exists a $K < 0$ such that for every flowline γ of $-\tilde{v}$ that starts in \tilde{N}_0 and ends in \tilde{N}_{-1} we have $\int_{\rho \circ \gamma} \omega \leq K$.*

Proof. We have $\rho^* \omega' = d\tilde{f}'$ and $\tilde{N}_k = (\tilde{f}')^{-1}(k \cdot b)$ with b as in 4.1. Since ω' is rational and \tilde{f}' has no critical points in \tilde{N}_0 there is a $t < 0$ such that $(\tilde{f}')^{-1}([t, 0])$ also contains no critical points. So if γ_p is a flowline of $-\tilde{v}$ with $\gamma_p(0) = p \in \tilde{N}_0$, there is a $t_p > 0$ which depends smoothly on p such that $\gamma_p(t_p) \in (\tilde{f}')^{-1}(\{t\})$. Now

$$\int_{\rho \circ \gamma_p|_{[0, t_p]}} \omega = \int_0^{t_p} \omega_{\rho \circ \gamma_p(s)}(-v(\rho \circ \gamma_p(s))) = - \int_0^{t_p} \omega_{\rho \circ \gamma_p(s)}(v(\rho \circ \gamma_p(s))) < 0$$

by Lemma 2.3. Since $\ker \xi'$ acts cocompactly on \tilde{N}_0 and the value of the integral depends smoothly on $p \in \tilde{N}_0$ there is $K < 0$ such that

$$\int_{\rho \circ \gamma_p|_{[0, t_p]}} \omega \leq K.$$

This K now works for the Lemma, since integrating over a longer flowline will just make the integral smaller. \square

Conclusion of the proof of 4.5: our flowline γ is the concatenation of flowlines $\gamma_1, \dots, \gamma_m$ to which Lemma 4.6 applies. Let $\tilde{f} : \tilde{M} \rightarrow \mathbb{R}$ satisfy $\rho^* \omega = d\tilde{f}$. Then we get

$$\xi(g) = \tilde{f}(gx) - \tilde{f}(x) = \int_{\gamma} d\tilde{f} = \sum_{l=1}^m \int_{\gamma_l} d\tilde{f} = \sum_{l=1}^m \int_{\gamma_l} \rho^* \omega = \sum_{l=1}^m \int_{\rho \circ \gamma_l} \omega \leq m \cdot K.$$

Therefore $\xi(g_1) + \dots + \xi(g_m) \leq m \cdot K$ for all $g_1 \in \text{supp } B_{j_{n_1}}, \dots, g_m \in \text{supp } B_{n_{m-1}j}$ which implies that B is ξ -regular. \square

Define $\mathcal{G}_0(\omega) = \{v \in \mathcal{G}(\omega) \mid v \text{ satisfies } (\mathfrak{A}\mathfrak{C})\}$. By the remarks above, $\mathcal{G}_0(\omega)$ is C^0 -dense in $\mathcal{G}(\omega)$.

Theorem 4.7. *Let ω be a Morse form on a smooth connected closed manifold M^n . Let $\xi : G \rightarrow \mathbb{R}$ be induced by ω and let $C_*^\Delta(\tilde{M})$ be the simplicial $\mathbb{Z}G$ complex coming from a smooth triangulation of M . For every $v \in \mathcal{G}_0(\omega)$ there is a natural chain homotopy equivalence $\varphi(v) : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \rightarrow C_*(\omega, v)$ given by (1) whose torsion $\tau(\varphi(v))$ lies in \overline{W} and satisfies*

$$\mathfrak{L}(\tau(\varphi(v))) = \eta(-v).$$

Proof. Since $v \in \mathcal{G}_0(\omega)$ we can form the $\mathbb{Z}G$ complex $\mathcal{C}(i-k)$ from the proof of Proposition 4.1 which is simple homotopy equivalent to $C_*^\Delta(\tilde{M})$. The matrices A_i are ξ -regular by Proposition 4.5, so the projection of $\widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} \mathcal{C}(i-k) \rightarrow \text{coker}(\text{id} \otimes i - k, \widehat{\mathbb{Z}G}_\xi)$ is a chain homotopy equivalence. We have already seen that the boundary homomorphisms of $C_*(\omega, v)$ and $C_*(\omega', v)$ are the same when viewed as matrices over $\widehat{\mathbb{Z}G}$. The same holds for $\text{coker}(\text{id} \otimes i - k, \widehat{\mathbb{Z}G}_\xi)$ and $\text{coker}(\text{id} \otimes i - k, \widehat{\mathbb{Z}G}_{\xi'})$. But since we identified $\text{coker}(\text{id} \otimes i - k, \widehat{\mathbb{Z}G}_{\xi'})$ with $C_*(\omega', v)$ we now get that $\text{coker}(\text{id} \otimes i - k, \widehat{\mathbb{Z}G}_\xi)$ is the same complex as $C_*(\omega, v)$. Also, if we use the triangulation from the proof of Proposition 4.1 we get that $\varphi(v)$ factors as $p \circ s$, where s is simple. Therefore

$$\tau(\varphi(v)) = \sum_{i=0}^{n-1} (-1)^{i+1} \tau(I - A_i) \in \overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi).$$

By Proposition 3.4 we have

$$\mathfrak{L}(\tau(\varphi(v))) = \sum_{i=0}^{n-1} (-1)^i \sum_{m=1}^{\infty} \frac{\varepsilon(\text{trace } A^m)}{m}.$$

By the proof of the Main Theorem in §5 of [20] the right hand side is exactly $\eta(-v)$. Of course, [20] only shows this in the rational case, but $\eta(-v)$ is independent of the homomorphism ξ when viewed as an element of $\widehat{\mathbb{Q}\Gamma}$. \square

5. COMPARISON WITH REIDEMEISTER TORSION

As mentioned in the introduction, for singular vector fields one of the first formulas to relate the torsion of the Novikov complex to a zeta function appeared in Hutchings and Lee [9, 10], a generalization appeared in Hutchings [7, 8] looking similar to Theorem 4.7, but using quite different methods. In this section we will relate these results, in fact we will show that Theorem 4.7 implies [8, Theorem B], at least for gradients satisfying condition $(\mathfrak{A}\mathfrak{C})$.

All these papers deal with commutative invariants only, so let \overline{M} be the universal abelian cover of M and $H = H_1(M)$ the covering transformation group. Let ω be a Morse form and v an ω -gradient satisfying the transversality condition. The Novikov complex in Hutchings [8] is given by $\overline{C}_*(\omega, v) = \widehat{\mathbb{Z}H}_\xi \otimes_{\widehat{\mathbb{Z}G}_\xi} C_*(\omega, v)$. Similarly $\widehat{\mathbb{Z}H}_\xi \otimes_{\mathbb{Z}H} C_*^\Delta(\overline{M}) = \widehat{\mathbb{Z}H}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M})$ and therefore a chain equivalence $\varphi(v) : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \rightarrow C_*(\omega, v)$ induces a chain equivalence $\bar{\varphi}(v) = \text{id} \otimes_{\widehat{\mathbb{Z}G}_\xi} \varphi(v) : \widehat{\mathbb{Z}H}_\xi \otimes_{\mathbb{Z}H} C_*^\Delta(\overline{M}) \rightarrow \overline{C}_*(\omega, v)$ with $\tau(\bar{\varphi}(v)) = \varepsilon_* \tau(\varphi(v)) \in \overline{K}_1^H(\widehat{\mathbb{Z}H}_\xi)$.

Let $Q\widehat{\mathbb{Z}H}_\xi$ be the localization of $\widehat{\mathbb{Z}H}_\xi$ along non-zero divisors. It is known that $Q\widehat{\mathbb{Z}H}_\xi$ is a finite direct product of fields, $Q\widehat{\mathbb{Z}H}_\xi = \bigoplus_{j=1}^k F_j$, see Hutchings [8, Lemma A.4] or Geoghegan and Nicas [6, Lemma 7.8]. Denote $p_j : Q\widehat{\mathbb{Z}H}_\xi \rightarrow F_j$ for the projection.

To define torsion in the sense of Hutchings [8], we need one more construction. Let R be a commutative ring with unit and U a subgroup of R^* , the group of units of R . We say that two elements of R are *equivalent*, $r \sim s$, if there exists a $u \in U$ such that $ru = s$. We denote by R/U the set of equivalence classes. The multiplication on R turns R/U into a semigroup which contains R^*/U as a subgroup.

Definition 5.1. [8, Def.A.1] Let F be a field and C_* a finite complex over F with a fixed basis and U a subgroup of F^* . Then the *Reidemeister torsion* of C_* is defined to be

$$\tau_R(C_*, U) = \begin{cases} 0 & \in F/U & \text{if } C_* \text{ is not acyclic} \\ \det(\tau(C_*))^{-1} & \in F^*/U \subset F/U & \text{if } C_* \text{ is acyclic} \end{cases}$$

Here $\tau(C_*) \in K_1(F)$ is Whitehead torsion.

We take the inverse of the determinant, because Hutchings [8] uses a different sign convention for torsion. When the group of units is clear, we will suppress it in the notation of the torsion.

Now $\pm H$ is a subgroup of $Q\widehat{\mathbb{Z}H}_\xi^*$. Denote $H_j = p_j(H) \subset F_j^*$ for $j = 1, \dots, k$. Then

$Q\widehat{\mathbb{Z}H}_\xi / \pm H = \bigoplus_{j=1}^k F_j / \pm H_j$. Hutchings [8, §1.5] defines

$$T_m = \sum_{j=1}^k \tau_R(F_j \otimes_{\widehat{\mathbb{Z}H}_\xi} \overline{C}_*(\omega, v)) \in Q\widehat{\mathbb{Z}H}_\xi / \pm H$$

and

$$T(\overline{M}) = \sum_{j=1}^k \tau_R(F_j \otimes_{\mathbb{Z}H} C_*^\Delta(\overline{M})) \in Q\widehat{\mathbb{Z}H}_\xi / \pm H.$$

Notice that $T(\overline{M})$, and hence T_m , can only be nonzero if $\chi(M) = 0$.³ These torsions are related by the zeta function of $-v$ which is defined as follows. Let

$\widehat{RG}_\xi^- = \{\lambda \in \widehat{RG}_\xi \mid \|\lambda\| < 1\}$. Similarly we get \widehat{RH}_ξ^- and $\widehat{R\Gamma}_\xi^-$. Define $\log : 1 + \widehat{\mathbb{Q}H}_\xi^- \rightarrow \widehat{\mathbb{Q}H}_\xi^-$ and $\exp : \widehat{\mathbb{Q}H}_\xi^- \rightarrow 1 + \widehat{\mathbb{Q}H}_\xi^-$ by

$$\log(1+a) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{a^m}{m} \quad \text{and} \quad \exp(a) = \sum_{m=0}^{\infty} \frac{a^m}{m!}.$$

It is readily seen that \log and \exp are well defined and mutually inverse to each other.

If $v \in \mathcal{G}(\omega)$, then $\eta(-v)$ is an element of $\widehat{\mathbb{Q}\Gamma}_\xi^-$ and we define the *zeta function* of $-v$ to be

$$\zeta(-v) = \exp(\varepsilon(\eta(-v))) \in 1 + \widehat{\mathbb{Q}H}_\xi^-.$$

Hutchings relates T_m and $T(\overline{M})$ by

Theorem 5.2. [8, Theorem B] *Let ω be a Morse form on the closed connected smooth manifold M , let $v \in \mathcal{G}(\omega)$ and let $\iota : 1 + \widehat{\mathbb{Q}H}_\xi^- \rightarrow Q\widehat{\mathbb{Z}H}_\xi / \pm H$ be given by inclusion and projection. Then $T_m \cdot \iota(\zeta(-v)) = T(\overline{M})$.*

We will show how to derive Theorem 5.2 from Theorem 4.7 for $v \in \mathcal{G}_0(\omega)$. It would be desirable to extend Theorem 4.7 to $v \in \mathcal{G}(\omega)$, possibly by the methods of [8].

So let $\varphi(v) : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \rightarrow C_*(\omega, v)$ be the chain homotopy equivalence from Theorem 4.7. As seen above, this induces a chain homotopy equivalence $\bar{\varphi}(v) : \widehat{\mathbb{Z}H}_\xi \otimes_{\mathbb{Z}H} C_*^\Delta(\overline{M}) \rightarrow \overline{C}_*(\omega, v)$. The homomorphism \mathfrak{L} actually gives a map $\mathfrak{L} : \overline{W} \rightarrow \widehat{\mathbb{Q}\Gamma}_\xi^-$. It is easy to see that the following diagram commutes.

$$\begin{array}{ccccccc} \overline{W} & \xrightarrow{\mathfrak{L}} & \widehat{\mathbb{Q}\Gamma}_\xi^- & \xrightarrow{\varepsilon} & \widehat{\mathbb{Q}H}_\xi^- & \xrightarrow{\exp} & 1 + \widehat{\mathbb{Q}H}_\xi^- \\ \downarrow & & & & & & \downarrow \\ \overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi) & \xrightarrow{\varepsilon_*} & \overline{K}_1^H(\widehat{\mathbb{Z}H}_\xi) & \xrightarrow{\det} & \widehat{\mathbb{Q}H}_\xi^* / \pm H & & \end{array}$$

³The Euler characteristic of the complex $F_j \otimes_{\mathbb{Z}H} C_*^\Delta(\overline{M})$ equals the Euler characteristic of M and since F_j is a field it can be calculated from the homology of that complex.

The last vertical arrow is inclusion of $1 + \widehat{\mathbb{Q}H_{\bar{\xi}}}^-$ into $\widehat{\mathbb{Q}H_{\bar{\xi}}}^*$ followed by projection. Hence we get $\det(\tau(\bar{\varphi}(v))) = \bar{\zeta}(-v) \in \widehat{\mathbb{Q}H_{\bar{\xi}}}^* / \pm H$. Look at the commutative diagram

$$(4) \quad \begin{array}{ccccc} \overline{K}_1^H(\widehat{\mathbb{Z}H_{\bar{\xi}}}) & \xrightarrow{\iota_*} & \overline{K}_1^H(Q\widehat{\mathbb{Z}H_{\bar{\xi}}}) & \xrightarrow{\simeq} & \bigoplus_{j=1}^k \overline{K}_1^{H_j}(F_j) \\ \downarrow \det & & \downarrow \det & & \downarrow \oplus \det \\ \widehat{\mathbb{Q}H_{\bar{\xi}}}^* / \pm H & \xrightarrow{\bar{\iota}} & Q\widehat{\mathbb{Z}H_{\bar{\xi}}}^* / \pm H & \xrightarrow{\simeq} & \bigoplus_{j=1}^k F_j^* / \pm H_j \end{array}$$

We will show that

$$(5) \quad p_j(T_m \cdot \iota(\zeta(-v))) = p_j(T(\overline{M})) \quad \text{for every } j = 1, \dots, k.$$

We have to compare $\det \circ (p_j)_* \circ \iota_* \tau(\bar{\varphi}(v))$ with $\tau_R(F_j \otimes \overline{C}(\omega, v))$ and $\tau_R(F_j \otimes C_*^\Delta(\overline{M}))$.

If $F_j \otimes \overline{C}(\omega, v)$ is acyclic, then so is $F_j \otimes C_*^\Delta(\overline{M})$, because $\text{id}_{F_j} \otimes \bar{\varphi}(v)$ is an equivalence of these complexes. Furthermore

$$\tau(\text{id}_{F_j} \otimes \bar{\varphi}(v)) = \tau(F_j \otimes \overline{C}(\omega, v)) - \tau(F_j \otimes C_*^\Delta(\overline{M})) \in \overline{K}_1^{H_j}(F_j).$$

So $\det(\tau(\text{id}_{F_j} \otimes \bar{\varphi}(v))) = \tau_R(F_j \otimes C_*^\Delta(\overline{M})) \cdot \tau_R(F_j \otimes \overline{C}(\omega, v))^{-1}$. This gives by (4) and Theorem 4.7

$$\tau_R(F_j \otimes \overline{C}(\omega, v)) \cdot p_j(\zeta(-v)) = \tau_R(F_j \otimes C_*^\Delta(\overline{M})).$$

If $F_j \otimes \overline{C}(\omega, v)$ is not acyclic, neither is $F_j \otimes C_*^\Delta(\overline{M})$ and (5) reduces to $0 = 0$. Hence we obtain the desired formula $T_m \cdot \iota(\zeta(-v)) = T(\overline{M})$.

Since the complexes $F_j \otimes \overline{C}(\omega, v)$ do not always have to be acyclic, Theorem 5.2 cannot recover the zeta function in general just from the torsion information. In particular, Theorem 5.2 contains no information for $\chi(M) \neq 0$. To see that we can get reasonable results for $\chi(M) \neq 0$ we have the following

Example 5.3. Let M be a closed surface of genus 2 and $f : M \rightarrow S^1$ a Morse map indicated by Figure 1, i.e. we take a projection $S^1 \times S^1 \rightarrow S^1$ and add a 1-handle which only gets

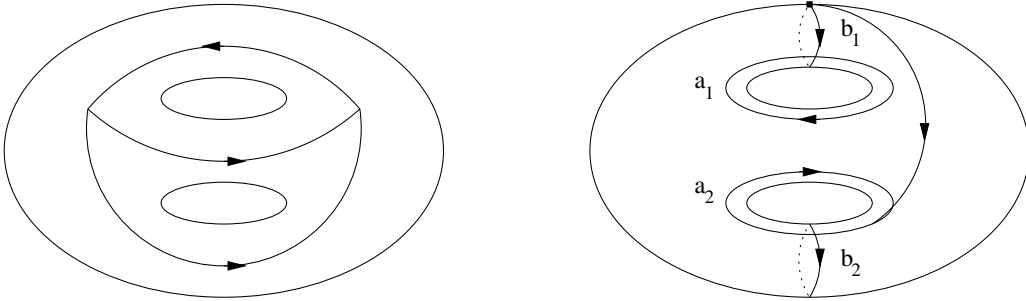


FIGURE 1.

mapped to one half of S^1 . This f has 2 critical points, both of index 1. With the loops

in Figure 1 we have $G = \pi_1(M) = \langle a_1, b_1, a_2, b_2 \mid [a_1, b_1][a_2, b_2] = 1 \rangle$. The homomorphism $\xi = f_\# : \pi_1(M) \rightarrow \mathbb{Z}$ is then given by $\xi(a_1) = -1$ and all other generators are sent to 0.

With the construction of 2.4 we get $N = S^1$ and M_N as in Figure 2. We can choose N so that it is the image of the loop b_1 , the basepoint being on the bottom. Put an f -gradient v on M so that the trajectories of $-v$ starting and ending at critical points are as in Figure 2.

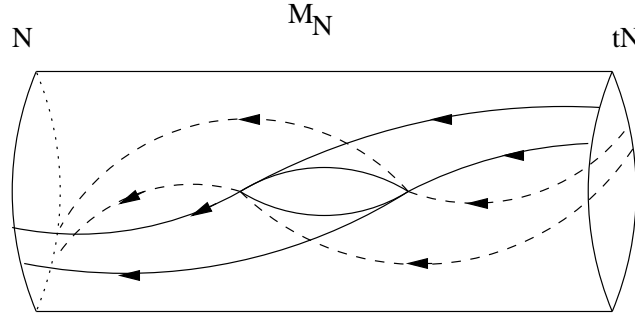


FIGURE 2.

We need v to satisfy condition (\mathfrak{C}') . The Morse map $\psi : N \rightarrow \mathbb{R}$ is chosen as the height function, so we have a minimum and a maximum. If the thickenings of the critical points on N are chosen to fill about half of the circle it is clear that we can find a v that satisfies (\mathfrak{C}') . Now we can also get a $v \in \mathcal{G}_0(f)$ with trajectories as in Figure 2.

Let $\varphi(v) : \widehat{\mathbb{Z}G}_\xi \otimes_{\mathbb{Z}G} C_*^\Delta(\tilde{M}) \rightarrow C_*(f, v)$ be the chain equivalence from Theorem 4.7. To calculate $\tau(\varphi(v))$ we have to look at the 1×1 matrices A_0 and A_1 that come from the negative gradient descent. All trajectories that start in tN and are not drawn in Figure 2 flow to N and cannot cross each other. To calculate A_0 notice that trajectories starting in the lower half of tN follow the loop that represents a_2a_1 up to conjugacy. Therefore $A_0 = (a_2a_1)$. The trajectories starting in the upper half of tN and ending in the upper half of N follow the loop a_1 up to conjugacy, so $A_1 = (a_1)$. Therefore

$$\tau(\varphi(v)) = \tau(1 - a_1) - \tau(1 - a_2a_1) = \tau((1 - a_1)(1 - a_2a_1)^{-1}) \in \overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi).$$

By Theorem 4.7 we get

$$\eta(-v) = \varepsilon(\log((1 - a_1)(1 - a_2a_1)^{-1})) \in \widehat{\mathbb{Q}\Gamma}_\xi$$

and

$$\zeta(-v) = (1 - [a_1])(1 - [a_2a_1])^{-1} \in \widehat{\mathbb{Q}H}_{\bar{\xi}}.$$

Remark 5.4. Notice that in Figure 2 the unstable manifolds of v intersect tN in the upper half of tN while the stable manifolds intersect N in the lower half of N . This allows v to satisfy (\mathfrak{C}') with the Morse map ψ on N . If we push the unstable manifolds down and the stable manifolds up in Figure 2, we get a different vector field w which also satisfies (\mathfrak{C}') ,

but with the Morse map $-\psi$. So if we want to calculate $\tau(\varphi(w))$ we have to interchange the roles of A_0 and A_1 which gives

$$\tau(\varphi(w)) = \tau(1 - a_2 a_1) - \tau(1 - a_1) = -\tau(\varphi(v)) \in \overline{K}_1^G(\widehat{\mathbb{Z}G}_\xi),$$

and

$$\zeta(-w) = (1 - [a_1])^{-1}(1 - [a_2 a_1]) \in \widehat{\mathbb{Q}H}_\xi.$$

We can interpret this the following way: By looking at Figure 2 we can expect two closed orbits, one on top of the cobordism, call it γ_1 , and one on the bottom, call it γ_2 . The conjugacy class represented by γ_1 is the class of a_1 while γ_2 represents the conjugacy class of $a_2 a_1$. Now $\varepsilon(\gamma_1) = -1$ and $\varepsilon(\gamma_2) = 1$ for the vector field v , but by passing to w the unstable and stable manifolds move and the signs change.

APPENDIX A. THE GEOMETRIC CHAIN HOMOTOPY EQUIVALENCE

The chain homotopy equivalence given by formula (1) has its counterpart in the exact case. The purpose of this appendix is to describe the properties in that case. An alternative approach can be found in Hutchings and Lee [9, §2.3], see also Schwarz [22, §4.2], but since we need the torsion of the equivalence, we give full proofs.

A.1. The relative Morse-Smale complex. Let $(W; M_0, M_1)$ be a compact cobordism, $f : W \rightarrow [a, b]$ a Morse function and v an f -gradient satisfying the transversality condition. A smooth triangulation Δ of W is said to be *adjusted to v* , if every k -simplex σ intersects the unstable manifolds $D_R(p)$ transversely for all critical points p of index $\geq k$. In particular, if p is a critical point of index k , a k -simplex σ intersects $D_R(p)$ in finitely many points. Using the orientations we can assign to every such point a sign. Given a regular covering space $q : \tilde{W} \rightarrow W$ we can use the covering transformation group G and liftings of critical points and simplices to assign an element $[\sigma : p] \in \mathbb{Z}G$ to the intersection and define a map

$$(6) \quad \begin{aligned} \varphi : C_*^\Delta(\tilde{W}, \tilde{M}_0) &\longrightarrow C_*^{MS}(\tilde{W}, \tilde{M}_0) \\ \sigma_k &\mapsto \sum_{p \in \text{crit}_k(f)} [\sigma : p] p \end{aligned}$$

Here $C_*^{MS}(\tilde{W}, \tilde{M}_0)$ is the Morse-Smale complex generated by the critical points of f . For $A \subset W$ we denote $\tilde{A} = q^{-1}(A)$. Before we show the existence of adjusted triangulations let us show that φ is indeed a chain map.

Lemma A.1. *φ is a chain map.*

Proof. There exists a filtration $M_0 = W_{-1} \subset W_0 \subset \dots \subset W_n = W$ of W such that W_i is a compact cobordism containing all critical points of index $\leq i$ and such that $C_k^{MS}(\tilde{W}, \tilde{M}_0) = H_k(\tilde{W}_k, \tilde{W}_{k-1})$ and the boundary homomorphism comes from the long exact sequence of the triple $(\tilde{W}_k, \tilde{W}_{k-1}, \tilde{W}_{k-2})$, see Milnor [13, §7]. Also, $C_k^\Delta(\tilde{W}, \tilde{M}_0) = H_k(\tilde{W}^{(k)}, \tilde{W}^{(k-1)})$, where $\tilde{W}^{(k)}$ denotes the k -skeleton of the triangulation. A simplex $\sigma_k \in C_k^\Delta(\tilde{W}, \tilde{M}_0)$ is represented by a map $f_\sigma : (\Delta^k, \partial\Delta^k) \rightarrow (\tilde{W}^{(k)}, \tilde{W}^{(k-1)})$. Let $\Phi : \tilde{W} \times \mathbb{R} \rightarrow \tilde{W}$ be induced by the flow of $-v$, where a flowline is supposed to stop once it hits the boundary. For $t \in \mathbb{R}$ let $\Phi_t = \Phi(\cdot, t)$.

Since Δ is adjusted to v there is a $t > 0$ such that $\Phi_t \circ f_\sigma$ maps Δ^k to \tilde{W}_k and $\partial\Delta^k$ to \tilde{W}_{k-1} . It follows from intersection theory that

$$\varphi(\sigma) = (\Phi_t \circ f_\sigma)_*[\Delta^k] \in H_k(\tilde{W}_k, \tilde{W}_{k-1}).$$

Furthermore this does not depend on t as long as t is large enough. A diagram chase gives that φ is a chain map. \square

Now we want to show the existence of adjusted triangulations. Let $\psi : W \rightarrow W$ be a diffeomorphism homotopic to the identity and Δ a smooth triangulation of W . Then $\psi\Delta$ is the triangulation of W where simplices are composed with ψ . The corresponding chain complexes can be identified by choosing a lifting $\tilde{\psi} : \tilde{W} \rightarrow \tilde{W}$.

So let Δ be any smooth triangulation and $\psi_{-1} = \text{id}_W$. We can adjust ψ_{-1} near the 0-skeleton so that 0-simplices intersect all unstable manifolds transversely. Since the boundary of W is transverse to the flow, we can leave it invariant. This way we get a diffeomorphism ψ_0 isotopic to the identity. Now assume ψ_{k-1} is isotopic to the identity and every j -simplex of $\psi_{k-1}\Delta$ with $j \leq k-1$ intersects the unstable manifolds transversely for critical points with index $\geq k-1$. We modify ψ_{k-1} on the k -skeleton so that k -simplices intersect $D_R(p)$ transversely for all p with index $\geq k$. Notice that for a k -simplex of $\psi_{k-1}\Delta$ this is already true for ψ_{k-1} near the boundary so we can leave the $(k-1)$ -skeleton fixed. This way we obtain ψ_k isotopic to the identity and we can proceed by induction.

Then $\psi_{n-1}\Delta$ is adjusted to v . Furthermore we can find an adjusted triangulation $\psi\Delta$ with ψ as close as we like to the identity. Moreover, compactness gives that if Δ is adjusted to v , so is $\psi\Delta$ for every ψ close enough to the identity. On the other hand, given a triangulation Δ and two diffeomorphisms ψ_1, ψ_2 homotopic to the identity such that $\psi_1\Delta$ and $\psi_2\Delta$ are adjusted to v , we get two chain maps φ_1 and φ_2 which can be different.

Lemma A.2. *The liftings can be chosen so that φ_1 and φ_2 are chain homotopic.*

Proof. Let $H' : W \times I \rightarrow W$ be a homotopy between φ_1 and φ_2 . As above we can change H' to a homotopy $H : W \times I \rightarrow W$ between φ_1 and φ_2 such that $H(\sigma \times I)$ intersects $D_R(p)$ transversely for all critical points p with $\text{ind} p \geq k+1$, where σ is a k -simplex. Describe the Morse-Smale complex as in the proof of Lemma A.1. Then we define $H_k : C_k^\Delta(\tilde{W}, \tilde{M}_0) \rightarrow C_{k+1}^{MS}(\tilde{W}, \tilde{M}_0)$ by $H_k(\sigma) = (-1)^k(\Phi_t \circ \tilde{H})_*[\sigma \times I] \in H_{k+1}(\tilde{W}_{k+1}, \tilde{W}_k)$. Here $t > 0$ is so large that $\Phi_t(\sigma \times \{0, 1\} \cup \partial\sigma \times I) \subset \tilde{W}_k$, $\Phi_t(\partial\sigma \times \{0, 1\}) \subset \tilde{W}_{k-1}$ and $\tilde{H} : \tilde{W} \times I \rightarrow \tilde{W}$ is a lifting of H . Use \tilde{H}_0 and \tilde{H}_1 to identify the triangulated chain complexes. Then H_k is the desired chain homotopy. \square

Notice that for a k -simplex σ and a disc $D_R(p)$ where $\text{ind} p = k+1$ $H(\sigma \times I) \cap D_R(p)$ is a finite set. So together with liftings and orientations we can write the chain homotopy as

$$H_k(\sigma) = \sum_{p \in \text{crit}_{k+1}(f)} [\sigma : p] p \quad \text{where } [\sigma : p] \in \mathbb{Z}G,$$

which is independent of the filtration and only involves intersection numbers.

Theorem A.3. *Let $f : W \rightarrow [a, b]$ be a Morse function, v an f -gradient satisfying the transversality condition, Δ a triangulation adjusted to v and $q : \tilde{W} \rightarrow W$ a regular covering space. Then $\varphi : C_*^\Delta(\tilde{W}, \tilde{M}_0) \rightarrow C_*^{MS}(\tilde{W}, \tilde{M}_0)$ given by (6) is a simple homotopy equivalence.*

Proof. Let Δ' be a subdivision of Δ . If $\psi\Delta'$ is adjusted to v , so is $\psi\Delta$. Moreover, the diagram

$$\begin{array}{ccc} C_*^{\psi\Delta}(\tilde{W}, \tilde{M}_0) & \xrightarrow{\text{sd}} & C_*^{\psi\Delta'}(\tilde{W}, \tilde{M}_0) \\ \varphi \searrow & & \swarrow \varphi \\ & C_*^{MS}(\tilde{W}, \tilde{M}_0) & \end{array}$$

commutes, where sd is subdivision, a simple homotopy equivalence. By Munkres [15, §10] it is good enough to show the theorem for a special smooth triangulation.

As in the proof of Lemma A.1 we have the filtration $M_0 = W_{-1} \subset W_0 \subset \dots \subset W_n = W$ where W_i is a compact cobordism containing the critical points of index $\leq i$. Choose a triangulation such that each W_i is a subcomplex for all $-1 \leq i \leq n$ and so that for each critical point p of index i the disc $D_i(p) = D_L(p) \cap (W_i - \text{int } W_{i-1})$ is a subcomplex. We set for $0 \leq k \leq n$ $C_*^{(k)} = C_*^\Delta(\tilde{W}_k, \tilde{M}_0)$. The complex $D_*^{(k)}$ is given by

$$D_i^{(k)} = \begin{cases} C_i^{MS}(\tilde{W}, \tilde{M}_0) & i \leq k \\ 0 & \text{otherwise} \end{cases}$$

The chain map φ induces maps $\varphi^{(k)} : C_*^{(k)} \rightarrow D_*^{(k)}$ and $\varphi^{(k,k-1)} : C_*^{(k)}/C_*^{(k-1)} \rightarrow D_*^{(k)}/D_*^{(k-1)}$. Since the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_*^{(k-1)} & \longrightarrow & C_*^{(k)} & \longrightarrow & C_*^{(k)}/C_*^{(k-1)} \longrightarrow 0 \\ & & \downarrow \varphi^{(k-1)} & & \downarrow \varphi^{(k)} & & \downarrow \varphi^{(k,k-1)} \\ 0 & \longrightarrow & D_*^{(k-1)} & \longrightarrow & D_*^{(k)} & \longrightarrow & D_*^{(k)}/D_*^{(k-1)} \longrightarrow 0 \end{array}$$

commutes, it suffices to show that each $\varphi^{(k,k-1)}$ is a simple homotopy equivalence to finish the proof.

Clearly $\varphi^{(k,k-1)}$ induces an isomorphism in homology, so it remains to show that it is simple. We set $D_i = \bigcup_{p \in \text{crit}_i(f)} D_i(p)$. Then the inclusion $i : C_*^\Delta(\tilde{W}_{i-1} \cup \tilde{D}_i, \tilde{W}_{i-1}) \rightarrow C_*^\Delta(\tilde{W}_i, \tilde{W}_{i-1})$ is

the inclusion of the core of the handles into the handles, hence a simple homotopy equivalence. Now $\varphi^{(k,k-1)} \circ i$ is a simple homotopy equivalence by Cohen [2, 18.3], since we can choose the lifts of D_i so that the matrices representing $\varphi^{(k,k-1)} \circ i$ and the boundary operators have only integer values. Therefore $\varphi^{(k,k-1)}$ is a simple homotopy equivalence. \square

Remark A.4. Pajitnov [16, Appendix A] describes a simple homotopy equivalence $\psi : C_*^{MS}(\tilde{W}, \tilde{M}_0) \rightarrow C_*^\Delta(\tilde{W}, \tilde{M}_0)$, where the triangulation is given by [16, Lm.A.8]. The liftings can be chosen so that $\varphi \circ \psi$ is the identity on the Morse-Smale complex, so φ and ψ are mutually inverse equivalences.

A.2. The absolute Morse-Smale complex. To calculate the absolute homology $H_*(\tilde{W})$ for a cobordism with nonempty boundary we need more technicalities. So if $f : W \rightarrow [a, b]$ is a Morse function on a compact cobordism and v an f -gradient, Pajitnov [18, 19] defines sets $D_\delta(\text{ind} \leq i; v)$ and $C_\delta(\text{ind} \leq i; v)$ for $\delta > 0$ and $0 \leq i \leq n$ which form filtrations of the cobordism. To simplify the notation we denote them by $D_\delta^i(v)$ and $C_\delta^i(v)$. Pajitnov [19, Df.4.2] now defines a condition (\mathfrak{C}) on the vector field which also involves Morse functions ϕ_i on M_i , ϕ_i -gradients u_i for $i = 0, 1$ and a $\delta > 0$.

Now we define a filtration of W by $W_i = C_\delta^i(u_0) \cup D_\delta^i(v)$. It follows from the methods of Pajitnov [18, §5] that $C_i^{MS}(\tilde{W}) = H_i(\tilde{W}_i, \tilde{W}_{i-1})$ gives a free $\mathbb{Z}G$ complex calculating $H_*(\tilde{W})$ with $C_i^{MS}(\tilde{W}) = C_i^{MS}(\tilde{M}_0) \oplus C_i^{MS}(\tilde{W}, \tilde{M}_0)$. We want to explicitly describe a chain homotopy equivalence between $C_*^\Delta(\tilde{W})$ and $C_*^{MS}(\tilde{W})$ based on A.1.

We need a triangulation Δ of W with subtriangulations Δ_i of M_i for $i = 0, 1$ with the properties 1.-4. described in Lemma A.5 below. Notice that for $\varepsilon > \delta > 0$ we have $C_\varepsilon^k(u_i) \subset C_\delta^k(u_i)$.

So if σ is a k -simplex of Δ we can flow it along $-v$ as in the proof of Lemma A.1 into W_k . This induces a chain map $\varphi : C_*^\Delta(\tilde{W}) \rightarrow C_*^{MS}(\tilde{W})$. Also $C_k^\Delta(\tilde{W}) = C_k^\Delta(\tilde{M}_0) \oplus C_k^\Delta(\tilde{W}, \tilde{M}_0)$ and $C_k^{MS}(\tilde{W}) = C_k^{MS}(\tilde{M}_0) \oplus C_k^{MS}(\tilde{W}, \tilde{M}_0)$. In this decomposition we have $\varphi = \begin{pmatrix} \varphi_1 & * \\ 0 & \varphi_2 \end{pmatrix}$, where $\varphi_1 : C_*^\Delta(\tilde{M}_0) \rightarrow C_*^{MS}(\tilde{M}_0)$ and $\varphi_2 : C_*^\Delta(\tilde{W}, \tilde{M}_0) \rightarrow C_*^{MS}(\tilde{W}, \tilde{M}_0)$ are the simple homotopy equivalences of A.1. Therefore φ is also a simple homotopy equivalence.

A.3. Existence of a nice triangulation. Let $(W; M_0, M_1)$ be a compact cobordism, $f : W \rightarrow [-\frac{1}{2}, n + \frac{1}{2}]$ an ordered Morse function and v an f -gradient satisfying the transversality condition and condition (\mathfrak{C}) of Pajitnov [19, §4]. Let $\phi_i : M_i \rightarrow \mathbb{R}$, u_i be the ϕ_i -gradients and $\delta > 0$ given through condition (\mathfrak{C}) .

Lemma A.5. *There exists a triangulation Δ of W having $M_0 \cup M_1$ as a subcomplex $\Delta_0 \cup \Delta_1$ with the following properties:*

1. Δ is adjusted to v .
2. For $i = 0, 1$ Δ_i is adjusted to u_i .
3. There is an $\varepsilon > \delta$ such that if σ is a k -simplex in Δ_i , then $\sigma \subset C_\varepsilon^k(u_i)$.
4. There is an $\varepsilon > \delta$ such that if $x \in W^{(k)}$ and the trajectory of $-v$ starting at x ends in $fl(x) \in M_0$, then $fl(x) \in C_\varepsilon^k(u_0)$.

Remark A.6. Notice that all conditions in Lemma A.5 are open in the sense that if Δ satisfies 1.-4., so does $\psi\Delta$, provided ψ is close enough to the identity in the smooth topology. Therefore a triangulation as needed in the proof of Proposition 4.1 also exists.

Proof of Lemma A.5. For $i = 0, 1$ choose triangulations Δ_i of M_i adjusted to u_i and let $\Phi^i : M_i \times \mathbb{R} \rightarrow M$ be the flow of $-u_i$. Then there is a $t > 0$ such that 3. is satisfied for $\Phi_t^i \Delta_i$. Extend $\Phi_t^0 \Delta_0 \cup \Phi_t^1 \Delta_1$ to a triangulation Δ of W . Choose a diffeomorphism ψ so close to the identity that $\psi\Delta$ is adjusted to v and 2. and 3. still hold. Modify this triangulation again so that if σ is a k -simplex and $x \in \sigma$ flows to $fl(x) \in M_0$, then $fl(x) \notin D_R(q; u_0)$ for critical points q of ϕ_0 with $\text{ind } q \geq k + 1$. Notice that $D_R(q, u_0)$ is at most $n - k - 2$

dimensional. By making only very small changes we now have a triangulation satisfying 1.-3. and by compactness the condition of 4. for some $\varepsilon > 0$, but not necessarily for $\varepsilon > \delta$. Rename this triangulation Δ and the two subcomplexes of the boundary Δ_0 and Δ_1 .

Notice that condition 4. already holds for Δ_0 and Δ_1 . This is trivial for Δ_0 and for Δ_1 it follows from condition (C). By continuity it also holds in a small collar neighborhood of the boundary. We can think of this collar as $f^{-1}([-\frac{1}{2}, -\frac{1}{2} + \eta) \cup (n + \frac{1}{2} - \eta, n + \frac{1}{2}])$ for some $\frac{1}{2} > \eta > 0$ and assume that there are no 0-simplices in $f^{-1}((n + \frac{1}{2} - \eta, n + \frac{1}{2}))$. Let $\xi : W \rightarrow [0, 1]$ be a smooth function which is 1 outside of the collar and 0 in a smaller collar. Let $\Phi : W \times \mathbb{R} \rightarrow W$ be the flow of the vector field $-\xi \cdot v$. There is a $T_1 > 0$ such that if σ is a k -simplex in Δ which does not meet M_1 , then $\Phi_t(\sigma) \subset W_k = f^{-1}([-\frac{1}{2}, k + \frac{1}{2}])$. Furthermore $\Phi_{T_1} \Delta$ satisfies 1.-3. and the same form of 4. as Δ does.

Let $V_k = f^{-1}(\{k - \frac{1}{2}\})$ for $k = 0, \dots, n$ and $U_k \subset f^{-1}[k - \frac{1}{2}, k)$ be diffeomorphic to $V_k \times [0, 1]$ with $(x, 0)$ corresponding to x and (x, t) lying on the same trajectory of v . Let $X_k = W_k - (\text{int } U_k \cup W_{k-1})$, then X_k is a compact cobordism. By changing f if necessary we can assume that $f|_{X_k}$ is a Morse function on this cobordism.

The Morse function $\phi_0 : M_0 \rightarrow \mathbb{R}$ is ordered so we can assume that $Y_k = \varphi_0^{-1}((-\infty, k + \frac{1}{2}])$ gives a filtration with $D_\delta^k(u_0) \subset Y_k \subset C_\delta^k(u_0)$ for $k = 0, \dots, n-1$. Let $\mu_k : M_0 \rightarrow [0, 1]$ be a smooth function with $\mu_k|_{Y_{k-1}} = 0$ and $\mu_k|_{M_0 - Y_k} = 1$ and define $\nu_k = \mu_k \cdot u_0$ and let $\Lambda_k : M_0 \times \mathbb{R} \rightarrow M_0$ be the flow of $-\nu_k$.

Define $\Theta_k : X_k \times \mathbb{R} \rightarrow X_k$ as follows. Let $(x, t) \in X_k \times \mathbb{R}$. If $x \in \bigcup_{\text{ind } p \leq k} D_R(p; v)$, let $\Theta_k(x, t) = x$. If not, the trajectory of $-v$ reaches $fl(x) \in M_0$. Let $fl_t(x) = \Lambda_k(fl(x), t)$. Now $fl_t(x)$ flows along v all the way back to $y \in f^{-1}(\{f(x)\})$. For if not, we have $fl_t(x) \in \bigcup_{\text{ind } p \leq k} D_L(p; v)$. By (C) we have $fl_t(x) \in Y_{k-1}$. But then $fl(x) = fl_t(x)$ because $\nu_k(fl_t(x)) = 0$ and then $fl_t(x)$ flows all the way back to x . This and the implicit function theorem give that $\Theta_k(x, t) = y$ is smooth on $X_k \times \mathbb{R}$. Similarly, a smooth homotopy λ_k between μ_k and μ_{k-1} with $\lambda_k|_{(M_0 - Y_k) \times [0, 1]} = 1$ and $\lambda_k|_{Y_{k-2} \times [0, 1]} = 0$ defines a smooth map $\Theta'_k : U_k \times \mathbb{R} \rightarrow U_k$ such that all Θ_k and Θ'_k define an isotopy $\Theta : W \times \mathbb{R} \rightarrow W$ with the following property : if $x \in W_k - \bigcup_{\text{ind } p \leq k} D_R(p; v)$ with $fl(x) \in M_0 - \bigcup_{\text{ind } q \geq k+1} D_R(q, u_0)$, then there is a $t_x > 0$ such that $\Theta(x, s) \in W_k - \bigcup_{\text{ind } p \leq k} D_R(p)$ and $fl(\Theta(x, s)) \in C_\delta^k(u_0)$ for all $s \geq t_x$. By compactness there is a $T_2 > 0$ such that $\Theta_{T_2} \Phi_{T_1} \Delta$ satisfies the Lemma. \square

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DEPARTMENT OF MATHEMATICS, SUNY BINGHAMTON, BINGHAMTON, NY 13902-6000

E-mail address: dirk@math.binghamton.edu